Continuity in κ in SLE$_\kappa$ theory using a constructive method and Rough Path Theory

In just 4 slides

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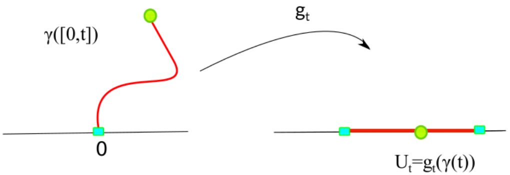
The Loewner Differential Equation



- The forward and backward Loewner Differential Equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad \dot{h}_t(z) = \frac{-2}{h_t(z) - U_t}.$$

- For $U_t = \sqrt{\kappa} B_t$, via $h_t(z) - \sqrt{\kappa} B_t = Z_t$, we obtain

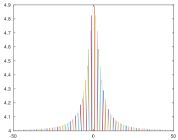
$$dZ_t = \frac{-2}{Z_t} dt - \sqrt{\kappa} dB_t.$$

- SLE curves $\gamma(t) := \lim_{y \to 0+} g_t^{-1}(iy + \sqrt{\kappa} B_t)$. [Rohde-Schramm $\kappa \neq 8$, Lawler-Schramm-Werner $\kappa = 8$.]
- **Continuity in κ of the SLE curves**: Rohde, Viklund, Wong, Friz, Yuan, Tran, Shekhar, etc.

SLE and RPT: Backward LDE as a RDE

$$dZ_t = \frac{-2/\kappa}{Z_t}dt + dB_t, \qquad Z_0 = z_0 \in \mathbb{H}.$$

- The backward LDE: $dZ_t = V(Z_t)dX_t$, where $V(Z) = (V_0(Z), V_1(Z))$ with $V_0(Z) = -\frac{2/\kappa}{Z}\frac{d}{dz}$ and $V_1(Z) = \frac{d}{dx}$ and $X_t = (t, B_t)$.



Theorem (Beliaev-Lyons-M.)

For $\epsilon > 0$, the backward LDE driven by $\sqrt{\kappa}B_t$ with $\kappa > 0$, started from z_0 with $|z_0| = \epsilon > 0$ is a well defined Rough Differential Equation that has a unique solution. This unique solution is a.s. continuous with respect to $\sqrt{\kappa}B_t$ in the p-variation topology, for $p \in (2, 3]$.

What about the continuity in κ of the *SLE* traces?

Constructively generating continuos in κ curves.

Theorem (Beliaev-Lyons-M.)

Let us fix $\kappa \in (0, 8/3)$ and a sequence $\kappa_j \to \kappa$, then for every T and almost every ω

$$\sup_{t \in [0,T]} |\gamma^{\kappa_j}(t) - \gamma^{\kappa}(t)| \to 0.$$

Moreover, for the curves $\gamma^{n,\kappa_j}(t)$ generated by the square-root interpolation of the drivers $\sqrt{\kappa_j} B_t$, we have that for almost every Brownian motion

$$\sup_{t \in [0,T]} |\gamma^{n,\kappa_j}(t) - \gamma^{n,\kappa}(t)| \to 0.$$

Remark

Result obtained by Friz, Tran and Yuan using different methods. We explore this direction differently by considering SLE_κ traces constructively for varying parameter $\kappa \in (0, 8/3)$.

- (FP) Rough Path lift of $X_t = (t, B_t)$, Universal Limit Theorem in Rough Path Theory.
- (SP) For $t_k = k/n$, $k = 0, \ldots, n$, we consider

$$U_t^n = \sqrt{n}(U_{t_{k+1}} - U_{t_k})\sqrt{t - t_k} + U_{t_k}, \quad t \in [t_k, t_{k+1}].$$

- For drivers of the form $U_t = c\sqrt{t} + d$, with c and d constants, the Loewner equation can be solved explicitly.
- The proof follows via:

$$|\gamma^{(1)}(t) - \gamma^{(2)}(t)| \leq |\gamma^{(1)}(t) - \gamma^{n,(2)}(t)| + |\gamma^{n,(2)}(t) - \gamma^{(2)}(t)|,$$

  where $\gamma^{n,(j)}$ is the trace obtained form interpolating with square-root terms the driver $\sqrt{\kappa_j}B_t$, $j = 1, 2$.
- **Difficulty:** Studying the continuity in a parameter in the setting of **singular dynamics**!
- **What can we learn?** A way to constructively show continuity in a singular dynamics settings. The use of RPT close to the singularity.