On the analytic extension of the Random Riemann Zeta Function for a probabilistic model of the primes

In just 2 slides

Stanislav Molchanov Vlad Margarint

https://margarintvlad.com/

- **Cramér model of quasi-primes** Let us call the numbers $n \geq 3$ *quasi-primes* with probability $\frac{1}{\log n}$ independently, for different n. We include 2 in the quasi-primes. Let $\{p_n(\omega)\}_{n\geq 1}$ be the increasing list of quasi-primes.
- In analogy with the Euler product, for $\Re(s) > 1$ consider the Random Euler product:

$$\zeta(s,\omega) = \prod_{n\geq 1} \frac{1}{1 - p_n(\omega)^{-s}}.$$

Theorem (M., Molchanov)

a) $\zeta(s,\omega)$ *has almost surely an analytic continuation on the region* $\{\Re(s) > 1/2\} \setminus (1/2, 1]$.
b) Let $s = \sigma + it$. *For any non-zero fixed* $t \in \mathbb{R} \setminus \{0\}$, *we have that* $\limsup_{\sigma\to 1/2^+} |\zeta(\sigma + it)| = \infty$, *almost surely.*

- **Proof idea:** Expand the Random Riemann Zeta.

$$\zeta(s,\omega) = exp\left(\sum_{n=1}^{\infty}\frac{1}{p_n^s(\omega)} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{p_n^{2s}(\omega)} + \dots\right).$$

  Higher-power terms are analytic already for $\Re(s) > \frac{1}{2}$. The key term reduces to a random series $\sum_{n\geq 3}\varepsilon_n n^{-s}$ with independent Bernoulli ε_n of mean $1/\ln n$, which we analyze.
- For part b): analysis of the real part of the Cramér Zeta function and use of probabilistic techniques (Borel-Cantelli Lemma, etc.)
- **What can we learn?**We analyzed the behavior of a **Random Riemann Zeta function** for one model of random primes. **One advantage of these random functions is that they are easier to generate on a computer. In contrast, to the best our knowledge, there is no direct formula to generate the sequence of the true prime numbers (mentioned to me also by Dr. Arindam Roy).**