Drivers, Hitting Times, and Weldings in Loewner Equation

In just 2 slides

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- A simple curve γ growing in the upper half-plane $\mathbb{H}$ (from the boundary) is encoded by a *driving function* $\xi(t)$ via the chordal Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \qquad g_0(z) = z.$$

- Let $S([0, T])$ denote the class of continuous drivers on $[0, T]$ that generate *simple* Loewner curves.
- Using the *reverse* flow

$$\partial_t h_t(z) = -\frac{2}{h_t(z) - \xi(t)}, \qquad h_0(z) = z,$$

boundary points $x \in \mathbb{R}$ are transported until they *hit* the driver.
- This defines the *hitting time function*

$$\tau(x; \xi) := \inf\{t \geq 0 : \; h_t(x) = \xi(t)\}.$$

- For simple curves, τ yields a *conformal welding* φ pairing left/right boundary points that are hitting the singularity at the same time:

$$\tau(x; \xi) = \tau(\varphi(x); \xi), \qquad x \leq 0 \leq \varphi(x),$$

on maximal intervals $[-a, 0]$ and $[0, b]$ where τ is finite.

- **Thm. 3.4: Drivers $\to$ hitting times:** If $\xi_n, \xi \in S([0,T])$ and $\|\xi_n - \xi\|_{\infty,[0,T]} \to 0$, then the maximal finiteness intervals converge $a_n \to a, \quad b_n \to b$, and $\tau(\cdot;\xi_n) \to \tau(\cdot;\xi)$ uniformly on any compact $[c,d] \subset (-a,b)$.
- **Lemma 3.6:** Consider starting points varying with n (no global modulus of continuity).
- **Thm. 3.8: Inverses of hitting times:** The inverses satisfy the sharp Lipschitz estimate with optimal constant 1.

$$\|\tau_{n,\pm}^{-1} - \tau_{\pm}^{-1}\|_{\infty,[0,T]} \le \|\xi_n - \xi\|_{\infty,[0,T]}.$$

- **Thm. 4.1: Drivers $\to$ weldings:** If $\xi_n \to \xi$ uniformly, then $\varphi_n \to \varphi$ uniformly on any $[-c,0] \subset (-a,0]$. Lemma 4.3: No global modulus of continuity.
- **Lemma 4.9:** Welding convergence does *not* imply hitting-time or driver convergence: $\varphi_n \to \varphi$ can hold while $\tau_n \not\to \tau$ and $\xi_n \not\to \xi$.
- **Difficulty**: Analyzing **singular dynamics in the deterministic setting**.
- **What can we learn?** Analytic (contrasting the probabilistic techniques) to study hitting times of the **singularity**, etc. for **singular deterministic dynamics**.