Law of the Schramm-Loewner Evolution (SLE) Tip
In just 3 slides

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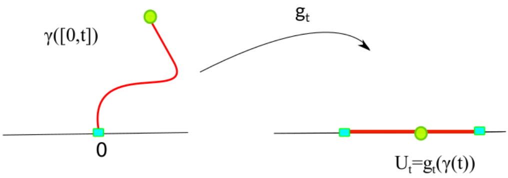
https://margarintvlad.com/



- The forward and backward Loewner Differential Equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \qquad \dot{h}_t(z) = \frac{-2}{h_t(z) - U_t}.$$

- For $U_t = \sqrt{\kappa}B_t$, via $h_t(z) - \sqrt{\kappa}B_t = Z_t = X_t + iY_t$, we obtain

$$dZ_t = \frac{-2}{Z_t}dt - \sqrt{\kappa}dB_t.$$

- SLE curves $\gamma_t := \lim_{y \to 0+} g_t^{-1}(iy + \sqrt{\kappa}B_t)$.[Rohde-Schramm $\kappa \neq 8$ and Lawler-Schramm-Werner $\kappa = 8$]

Law of the SLE tip at a fixed time

Theorem (Butkovsky-M.-Yuan)

The random vector $(\mathrm{Re}\,(\gamma_1)\,, \mathrm{Im}\,(\gamma_1))$ *has a density which is the unique solution in the class of probability densities of the following PDE:*

$$\frac{\kappa}{2}\partial_{xx}^2\psi + \left(\frac{1}{2}x + \frac{2x}{x^2+y^2}\right)\partial_x\psi + \left(\frac{1}{2}y - \frac{2y}{x^2+y^2}\right)\partial_y\psi + \left(1 + \frac{4\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}\right)\psi = 0,$$

where $x \in \mathbb{R}, y \in (0,\infty)$. *Furthermore,* ψ *is strictly positive in* $\mathbb{R}\times(0,2), \psi \equiv 0$ *on* $\mathbb{R}\times[2,\infty)$, *and* $\psi(x,y) = \psi(-x,y)$ *for* $x \in \mathbb{R}, y > 0$.

One of our initial motivations was to obtain the marginal law of $\alpha = \arg\gamma(1)$. **We conjecture** $\alpha^{8/\kappa}$ **as** $\alpha \searrow 0$**.**

- We study this law via the unique invariant measure for the process $\left(\widehat{X}_t, \widehat{U}_t\right)_{t\geqslant 0}$ obtained from the backward LDE via a change of coordinates

$$d\widehat{X}_t = \left(-\frac{1}{2}\widehat{X}_t - \frac{2\widehat{X}_t}{\widehat{X}_t^2 + e^{2\widehat{U}_t}}\right) dt + \sqrt{\kappa} d\widehat{B}_t,$$

$$d\widehat{U}_t = \left(-\frac{1}{2} + \frac{2}{\widehat{X}_t^2 + e^{2\widehat{U}_t}}\right) dt$$

  That is,

$$\left(\widehat{X}_t, \widehat{U}_t\right) \to \left(\mathrm{Re}\left(\gamma_1\right), \log\left(\mathrm{Im}\left(\gamma_1\right)\right)\right) \text{ in law as } t \to \infty.$$

- We show that the density p of $\pi := \mathrm{Law}\left(\mathrm{Re}\left(\gamma_1\right), \log\left(\mathrm{Im}\left(\gamma_1\right)\right)\right)$ is the unique solution in the class of densities of a certain Fokker-Planck-Kolmogorov (FPK) equation.
- To study the support, we use, between others, Harnack Inequality.
- **What can we learn?**We can characterize the density of a law of the SLE tip at a fixed time as a unique solution of a certain FPK equation using modern techniques involving criteria on coefficients, etc.