SLE theory, Rough Paths, Numerical Analysis, Machine Learning

An overview in just 9 slides

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joint work with T. Lyons J. Foster; J. Chen; N. Shrotri

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Planar Statistical Physics. What is the scaling limit?

- A sample of a planar Loop-Erased Random Walk:

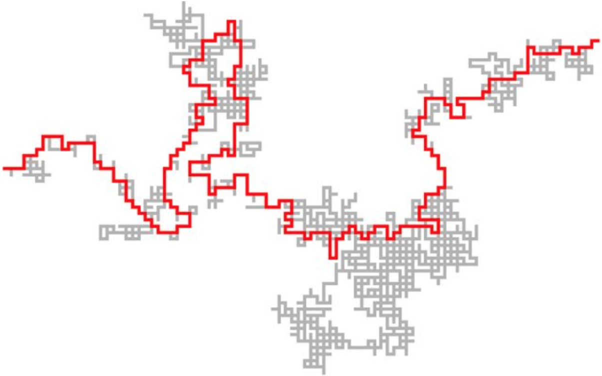
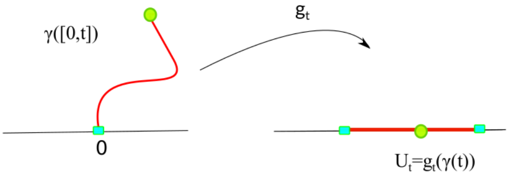


Figure: Credit to E. Peltola, A. Karrila, K. Kytola.

The Loewner Differential Equation

- For a non-self crossing curve $\gamma(t) : [0,\infty) \to \bar{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(\infty) = \infty$, we consider the simply connected domain $\mathbb{H} \setminus \gamma([0,t])$.



- The forward and backward Loewner Differential Equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \qquad \dot{h}_t(z) = \frac{-2}{h_t(z) - U_t}.$$

- For $U_t = \sqrt{\kappa} B_t$, via $h_t(z) - \sqrt{\kappa} B_t = Z_t$, we obtain

$$dZ_t = \frac{-2}{Z_t} dt - \sqrt{\kappa} dB_t.$$

- SLE curves $\gamma(t) := \lim_{y \to 0+} g_t^{-1}(iy + \sqrt{\kappa} B_t)$.

Backward LDE and Stochastic Taylor Approximations

$$dZ_t = \frac{-2/\kappa}{Z_t}dt + dB_t, \qquad Z_0 = z_0 \in \mathbb{H}.$$

- As before, $dZ_t = V(Z_t)dX_t$, where $V(Z) = (V_0(Z), V_1(Z))$, with $V_0(Z) = -\frac{2/\kappa}{Z}\frac{d}{dz}$ and $V_1(Z) = \frac{d}{dx}$ and $X_t = (t, B_t)$.
- We study

$$\mathcal{E}^r_{(V)}(Z_0, \mathbf{X}_{0,t}) = \sum_{I:deg(I)\le r} V_{i_1}\cdots V_{i_k}\mathbf{Id}(Z_0)\mathbf{X}^{i_1,\cdots,i_k}_{0,t},$$

  where

$$(s,t) \to \mathbf{X}_{s,t} = (1, X^1_{s,t}, \cdots, X^n_{s,t}, \cdots)$$

  is the signature of the path X_t and $\deg_{(p,q)}(I) := \frac{n}{p} + \frac{m}{q}$, with n-dB_t entries and m-dt entries.
- We estimate

$$R_r(t, Z_t, \mathbf{X}) = Z_t - \mathcal{E}^r_{(V)}(Z_0, \mathbf{X}_{0,t}).$$

Backward LDE and Stochastic Taylor Approximations

Lemma

For $Z_0 = \epsilon i$ *with* $\epsilon > 0$*, we have that* $|V_{i_1} \cdots V_{i_r} \mathbf{Id}(Z_0)| = O\left(\frac{1}{\epsilon^{2m-1+n}}\right)$.

Theorem (Foster-Lyons-M.)

The one-step r*-truncated Taylor approximation for the backward Loewner differential equation started from* $Z_0 = \epsilon i$*, with* $\epsilon > 0$ *sufficiently small, admits an* $O(\epsilon)$ *error in an* $L^2(\mathbb{P})$ *sense over the time horizon* $0 \le t \le \epsilon^2$*. In other words, for* C *a finite constant that depends only on* r*, we have*

$$\left\| Z_t - \mathcal{E}^r_{(V)}(Z_0, \mathbf{X}_{0,t}) \right\|_{L^2(\mathbb{P})} \le C\,\epsilon, \quad \forall t \in \left[0, \epsilon^2\right].$$

Remark: Note that on a time horizon of $\epsilon^{2-\delta}$

$$\left\| V_{i_1} \cdots V_{i_l} \mathbf{Id}(Z_0) \int_{0 < s_1 < \cdots < s_l < \epsilon^{2-\delta}} dX^{i_1}_{s_1} \cdots dX^{i_l}_{s_l} \right\|_{L^2(\mathbb{P})} = O\left(\epsilon^{1-\delta \deg_{(1,2)}(I)}\right).$$

We construct a numerical solution $\left(\widehat{Z}_{t_k}\right)_{0\le k\le N}$ for each $k \in [0\dots N-1]$, defining $\widehat{Z}_{t_{k+1}}$ using the formula

$$\widehat{Z}_{t_{k+1}} := \sqrt{\left(\sqrt{\widehat{Z}_{t_k}^2 - 2h_k} + \sqrt{\kappa}\, B_{t_k,t_{k+1}}\right)^2 - 2h_k},$$

where $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ is a partition of $[0, T]$ and $h_k := t_{k+1} - t_k$.

Proposition

Let Z is the true solution of backward LDE with the initial condition $Z_0 = \widehat{Z}_0$. Then for $k \ge 0$,

$$\mathbb{E}\left[\widehat{Z}_{t_k}^2\right] = \mathbb{E}\left[Z_{t_k}^2\right], \qquad \mathbb{E}\left[\widehat{Z}_{t_k}^4\right] = \mathbb{E}\left[Z_{t_k}^4\right].$$

- **Adaptive** step-size simulations.

Simulations of the SLE traces with the NV method

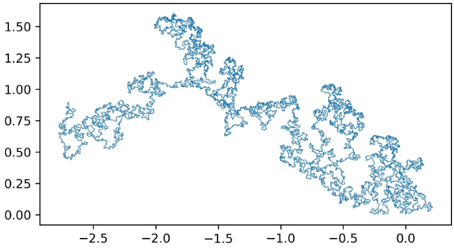


Figure: Simulation of the $SLE(6)$ trace using the Ninomiya-Victoir scheme. Credits to James Foster.

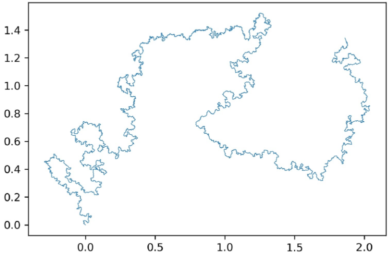


Figure: Simulation of the $SLE(8/3)$ trace using the Ninomiya-Victoir scheme. Credits to James Foster.

Order of convergence of the NV Scheme

Theorem (Chen-M.)

Let $\gamma(t)$ be the SLE trace and let $\gamma^n(t)$ be the approximated trace using the NV scheme. There exist two explicit non-increasing functions $\varphi_i : \mathbb{N} \to \mathbb{R}_+$ so that $\lim_{n\to\infty} \varphi_i(n) = 0^+$ with $i = 1, 2$, and

$$\mathbb{P}\left(\|\gamma(t) - \gamma^n(t)\|_{[0,1],\infty} \leq \varphi_1(n)\right) \geq 1 - \varphi_2(n).$$

- **Other types of convergences** studied also in the work with J. Chen.
- Loewner chains driven by Noise-Reinforced BM and by Fractional BM.
  1) $B_t^p := t^p \int_0^t s^{-p}\, dB_s$, $\mathbb{E}[B_s^p B_t^p] = (1-2p)^{-1} s^{1-p} t^p$, $\forall\, 0 \leq s \leq t$.
  2) $B_t^H := \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \int_0^t (t-s)^{H-\frac{1}{2}}\, dB_s$, with covariance
  $$\mathbb{E}[B_s^H B_t^H] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right).$$

- **Difficulty: The analysis is involved; singular dynamics involved**: Considered several events in which one uses properties of **the BM driver, SLE map estimates, and the exact definition of the NV scheme**. Along these, the functions from the theorem are tracked.
- With N. Shrotri, we trained NN (Deep Learning) using the NV scheme:

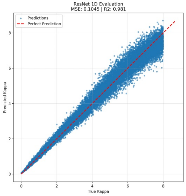


Figure 4: True vs. Predicted κ using the 1D ResNet. The predictions lie tightly along the diagona.

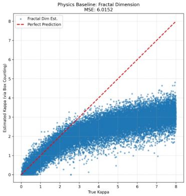


Figure 5: True vs. Predicted κ using the Hausdorff dimension baseline. The high variance highlights the difficulty of extracting κ from discrete, finite-length realizations via box-counting.

- **What can we learn?** The analysis of the other numerical methods, lead to NV scheme, adaptive step-sizes, and analysis of the convergence and new simulations using that scheme for other involved drivers, as well as ML and NN training with it.