Scaling Limits of Disorder Relevant Non-Binary Spin Systems

In Just 3 slides

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- **Lattice setup:** for a bounded open domain $\Omega \subset \mathbb{R}^d$, $\Omega_\delta := \Omega \cap (\delta\mathbb{Z})^d, \delta > 0$. A (real-valued) spin $\sigma_x \in \mathbb{R}$ sits at each $x \in \Omega_\delta$; the **reference** law is $P^{\mathrm{ref}}_{\Omega_\delta}$ with expectation $E^{\mathrm{ref}}_{\Omega_\delta}$.
- **Disorder (random field):** i.i.d. $(\omega_x)_{x\in\Omega_\delta}$ with $E[\omega_x] = 0$, $E[\omega_x^2] = 1$, and log-mgf

$$\phi(\lambda) := \log E\left[e^{\lambda\omega_x}\right] \quad \text{finite for } |\lambda| < \lambda_0.$$

- **Random-field perturbation:** Gibbs measure

$$P^{\omega}_{\Omega_\delta;\lambda}(d\sigma) := \frac{\exp\left(\sum_{x\in\Omega_\delta} \lambda\,\omega_x\sigma_x\right)}{Z^{\omega}_{\Omega_\delta;\lambda}}\, P^{\mathrm{ref}}_{\Omega_\delta}(d\sigma),$$

  with partition function

$$Z^{\omega}_{\Omega_\delta;\lambda} := E^{\mathrm{ref}}_{\Omega_\delta}\left[\exp\left(\sum_{x\in\Omega_\delta} \lambda\,\omega_x\sigma_x\right)\right].$$

- **Goal (intermediate disorder):** choose $\lambda = \lambda_\delta \downarrow 0$ so that (after centering/scaling) $Z^{\omega}_{\Omega_\delta;\lambda_\delta}$ has a **non-trivial continuum limit**.

- Study the modified (normalized) partition function

$$\widehat{Z}^{\omega}_{\Omega_\delta;\lambda} := E^{\mathrm{ref}}_{\Omega_\delta}\left[\exp\left(\sum_{x\in\Omega_\delta}\left(\lambda\omega_x\sigma_x - \phi(\lambda\sigma_x)\right)\right)\right],$$

so that $E[\widehat{Z}^{\omega}_{\Omega_\delta;\lambda}] = 1$. When $(\omega_x)_{x\in\Omega_\delta}$, i.i.d. standard Gaussians, this corresponds to Wick exponential: $e^{\lambda\omega_x\sigma_x} \mapsto :e^{\lambda\omega_x\sigma_x} := e^{\lambda\omega_x\sigma_x - \lambda^2\sigma_x^2/2}$.

- **Assumptions on $P^{\mathrm{ref}}_{\Omega_\delta}$:**

(A0) **Bounded spins:** $\exists\, K > 0$ s.t. $P^{\mathrm{ref}}_{\Omega_\delta}(\sigma_x \in [-K, K]) = 1$ for all x, δ.

(A1) **Limit of correlations:** $\exists\, \gamma > 0$ such that for each $k \in \mathbb{N}$,

$$\psi_\delta(x_1, \ldots, x_k) := \mathbf{1}_{\{x_i^\delta \neq x_j^\delta\ \forall i\neq j\}}\, \delta^{-k\gamma}\, E^{\mathrm{ref}}_{\Omega_\delta}\left[\sigma_{x_1^\delta}\cdots\sigma_{x_k^\delta}\right] \longrightarrow \psi_0 \quad \text{in } L^2(\Omega^k).$$

(A2) **Uniform tail control:** for any $\widehat{\lambda} > 0$,

$$\lim_{M\to\infty} \limsup_{\delta\downarrow 0} \sum_{k=M+1}^{\infty} \frac{\widehat{\lambda}^{2k}}{k!} \|\psi_\delta\|_{L^2(\Omega^k)}^2 = 0.$$

(A3) **Extra assumption useful to control higher-order terms:** for distinct $x_1^\delta, \ldots, x_k^\delta \in \Omega_\delta$ and $r_i \in \mathbb{N}$, letting $(r_i)_2 := r_i \bmod 2 \in \{0,1\}$, there exists $C \geq 1$ such that

$$\left| E_{\Omega_\delta}^{\mathrm{ref}} \left[\prod_{i=1}^{k} \sigma_{x_i^\delta}^{r_i} \right] \right| \leq C^{\sum_{i=1}^{k}(r_i-(r_i)_2)} \left| E_{\Omega_\delta}^{\mathrm{ref}} \left[\prod_{i=1}^{k} \sigma_{x_i^\delta}^{(r_i)_2} \right] \right|.$$

- **Intermediate disorder scaling:** for $\gamma < \frac{d}{4}$ and $\widehat{\lambda} > 0$, set $\lambda_\delta := \widehat{\lambda}\, \delta^{\frac{d}{2}-\gamma}$.
- **Theorem (Li, M., Sun)** As $\delta \downarrow 0$, in distribution,

$$\widehat{Z}_{\Omega_\delta;\lambda_\delta}^{\omega} \;\Rightarrow\; Z_{\Omega,\widehat{\lambda}}^{W} := 1 + \sum_{k=1}^{\infty} \frac{\widehat{\lambda}^k}{k!} \int_{\Omega^k} \psi_0(x_1,\ldots,x_k)\, dW(x_1)\cdots dW(x_k),$$

  where W is white noise on $\mathbb{R}^d$, and the series converges in L^2.
- **What can we learn?** A way to study the scaling limits beyond the previously studied binary spins models.