A bridge between Random Matrix Theory and Schramm-Loewner Evolutions Theory

In just 5 slides

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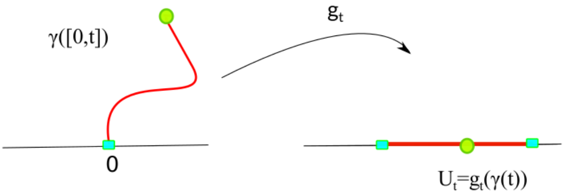
Loewner Differential Equation and SLE



- The maps satisfy the Loewner Differential Equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}\,, \qquad g_0(z) = z\,.$$

- SLE: $U_t = \sqrt{\kappa} B_t$.
- **Brownian Motion** $\rightarrow$ **Dyson Brownian Motion**(DBM), $\beta = 8/\kappa$.

$$d\lambda_t^i = \frac{\sqrt{2}}{\sqrt{N\beta}} dB_t^i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_t^j - \lambda_t^i}, i = 1, \cdots, N.$$

We consider

$$\partial_t g_t(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t(z) - \lambda_t^j}.$$

Theorem (Campbell-Luh-M.)

Let $\beta = 1$ or $\beta = 2$, and let us consider Dyson Brownian motion beginning at the origin. Let K_T be the multiple SLE hull at time $T > 0$. Then, for any $\varepsilon > 0$, for the multiple SLE maps for N curves, we have that

$$\sup_{t\in[0,T], z\in G} \left| g_t^N(z) - g_t^\infty(z) \right| = O\left(\frac{1}{N^{1/3-\varepsilon}} \right),$$

with overwhelming probability, for a given $G \subset \mathbb{H} \backslash K_T$.

- Uses modern RMT techniques such as Stieltjes transforms, self-consistent equations, etc.

Idea of the proof of the result

First, Del Monaco and Schleissinger:

$$\frac{\partial}{\partial t} g_t^{\infty}(z) = M_t^{\infty}(g_t(z)), \quad g_0(z) = z,$$

where M_t^{∞} is a solution to the complex Burgers equation

$$\begin{cases} \frac{\partial M_t^{\infty}(z)}{\partial t} = -2M_t^{\infty}(z)\frac{\partial M_t^{\infty}(z)}{\partial z}, t > 0 \\ M_0^{\infty}(z) = \int_{\mathbb{R}} \frac{2}{z-x} d\mu_0(x) \end{cases}$$

- Let $M_t^N(\bullet) = \frac{1}{N}\sum_{j=1}^{N} \frac{2}{\bullet - \lambda_t^j}$.
- Let us consider the time interval $[0,1]$ and a uniform partition with $t_k = \frac{k}{n}, k = 0, 1, \ldots, n$. Let $t \in (t_1, t_2)$.
- The proof is based on controlling $\sup_{\bullet \in G} \left| M_t^{\infty}(\bullet) - M_t^N(\bullet) \right| \leq \sup_{\bullet \in G} \left| M_t^{\infty}(\bullet) - M_{t_1}^{\infty}(\bullet) \right| + \sup_{\bullet \in G} \left| M_{t_1}^{\infty}(\bullet) - M_{t_1}^N(\bullet) \right| + \sup_{\bullet \in G} \left| M_{t_1}^N(\bullet) - M_t^N(\bullet) \right|$, for $G \subset \mathbb{H}$.

Idea of the proof of the result

- Let $\beta = 1$. We have that $M_t^N(\bullet) = \frac{-2}{N} \operatorname{tr}\left(A_t - \bullet \cdot I\right)^{-1}$, where $A_t = \sqrt{t}A$, with A a GOE.
- For M_t^∞, we have $M_t^\infty(\bullet) - S^N\left(z - 2tM_t^\infty(\bullet)\right) = 0$, with $S^N(\bullet) = -\frac{2}{N} \operatorname{tr} Q(\bullet)$, where $Q(\bullet)$ is a certain resolvent matrix.
- By a union bound, we have that, for any $\epsilon > 0$

$$
\begin{aligned}
&\mathbb{P}\left(\bigcup_{t_k} \left| M_{t_k}^\infty(\bullet) - M_{t_k}^N(\bullet)\right| = \Omega\left(\frac{C}{N^{1/3-\epsilon}}\right)\right) \\
&\leq \sum_{k=1}^{n} \mathbb{P}\left(\left| M_{t_k}^\infty(\bullet) - M_{t_k}^N(\bullet)\right| = \Omega\left(\frac{C}{N^{1/3-\epsilon}}\right)\right) \\
&\leq n e^{-CN}.
\end{aligned}
$$

- Note $g = \Omega(f) \leftrightarrow f = O(g)$.

Multiple SLE and Random Matrix Theory

$$\partial_t g_t(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t(z) - \lambda_t^j}.$$

- Rewrite RHS of Multiple Loewner equation as $M_t^N(\bullet) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{\bullet - \lambda_t^j}$.
- Recognize Stieltjes transform for a given choice of measure.
- **Let $\beta = 1$. We have that $M_t^N(\bullet) = \frac{-2}{N} \operatorname{tr} \left(A_t - \bullet \cdot I\right)^{-1}$, where $A_t = \sqrt{t}A$, with A a GOE**
- RHS of Loewner can be recovered as a Random Matrix Theory object.
- **The critical parameters $\kappa = 4$ and $\kappa = 8$ in the SLE theory correspond to the 'nice' $\beta = 2$ and $\beta = 1$ parameters in the Random Matrix Theory.**