Using results on real Bessel processes in the study of SLE

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Overview

1. The setting
2. Boundary behaviour of (backward) Bessel process and phase transition of the backward Loewner flow
   - Bessel processes of low and negative dimensions
   - Uniqueness and non-uniqueness of backward Loewner flow started from the origin
3. (Backward) Bessel processes and the conformal welding homeomorphisms
4. (Forward) Bessel processes and the control of nullsets when varying κ – towards the continuity in κ of the traces – in progress, writing phase.

The setting

- Consider the probability space $(\Omega, \mathcal{F}_t, \mathbb{P}_B)$, with the Wiener measure associated with the Brownian paths.

- The Bessel SDE has strong solutions, thus we build the Bessel processes as functions of the driver and we consider a coupling of these processes with the same Brownian motion.

- For fixed index i, we take

$$f_{\kappa_i}^{x_0} \ : (\Omega, \mathcal{F}_t, \mathbb{P}_B) \to (\Omega, \mathcal{F}'_t, f_{\kappa_i}^{x_0} * \mathbb{P}_B)$$

$$f_{\kappa_i}^{x_0}(B_t(\omega)) = X_t^{x_0,\kappa_i}(\omega)\,,$$

with $X_t^{x_0,\kappa_i}$ a Bessel process of dimension $d(\kappa_i)$ starting from $x_0 > 0$.

The setting

- Let us consider, the family of sets

$$\Omega_{\kappa_i} := \{\omega \in \Omega | \forall t \in [0,1], \exists \ \gamma_t^{(\kappa_i)} := \lim_{y \to 0+} g_t^{-1}(iy)\}$$

with $i \in \mathbb{R}$ a varying index with $\kappa_{i_0} = 0+$ and $\kappa_{i_\infty} = +\infty$, $k_i \neq 8$.

As well as,

$\tilde{\Omega}_{\kappa_i}^{x_0} := \{\omega \in \Omega |\ X_t^{x_0,\kappa_i}(\omega)$ is a real Bessel process of dimension $d(\kappa_i), x_0 > 0\}$

- A natural rigidity: we extend the maps to the boundary, then whenever the trace exists a.s., we have on the same full measure set a collection of Bessel processes on the real line indexed by starting points and driven by the same Brownian paths.

Bessel processes of low and negative dimension

- Considering the extensions to the boundary of the backward Loewner maps $h_t(z)$, we obtain that $X_t(x) := \frac{h_t(x\sqrt{\kappa})-\sqrt{\kappa}B_t}{\sqrt{\kappa}}$ solves

$$dX_t = \frac{-2/\kappa}{X_t}dt + dB_t\,, \quad X_0 = x.$$

- We have $-2/\kappa = (d-1)/2$, i.e. $d = 1 - 4/\kappa$. Thus, $d \leq 0$ for $\kappa \leq 4$ and $d > 0$ for $\kappa > 4$.

- For $d \in (0, 1]$, we can define uniquely Bessel process that is instantaneously reflecting at $\{0\}$. In this regime, the process is not a semimartingale and satisfies a principal value version of the equation.

Applications to SLE

- Let $\phi(n)$ be a subpower function. Let us introduce the following 'boxes' in $\mathbb{H}$:

$$A_{n,c,\phi} = \{x + iy \in \mathbb{H} : |x| \leq \frac{\phi(n)}{\sqrt{n}}, \frac{1}{\sqrt{n}\phi(n)} \leq y \leq \frac{c}{\sqrt{n}}\}.$$

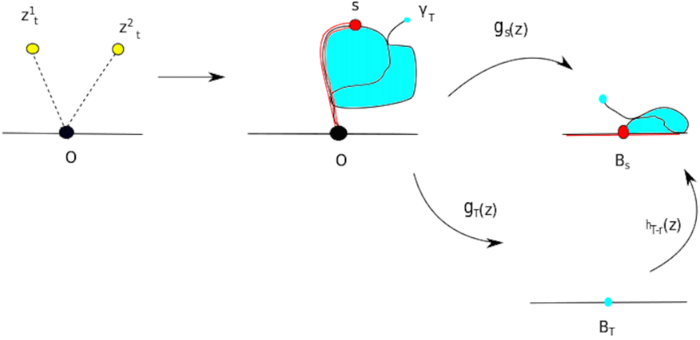
Proposition (Part 1)

Let $\kappa \in (0, 4]$. Then a.s. for all fixed $t \in [0, 1]$, there is a unique solution for the backward Loewner differential equation started from the origin in $\mathbb{H}$. For defining the solution, let us consider a sequence of points $(z_n)_{n\in\mathbb{N}} \in A_{n,c,\phi}$, such that $z_n \to 0+$ as $n \to \infty$. Then, for all fixed $t > 0$, a way to describe the solution is

$$h_t(0+) := \lim_{n\to\infty} h_t(z_n).$$

Elements of the proof: The uniqueness for $\kappa \leq 4$

- We show that for $\kappa \leq 4$, *zero* can not be mapped under the backward Loewner flow to two different points $z_1(t)$ and $z_2(t)$ in $\mathbb{H}$.
- By de-slitting conveniently for an intermediate time, we obtain an intersection of the trace with the real axis, obtaining a contradiction with $\{0\}$ is absorbing boundary.



Elements of the proof: The uniqueness for $\kappa \leq 4$

- In order to define the solution, we consider starting points $(z_n)_{n\in\mathbb{N}} \in A_{n,c,\phi}$, by combining universal results about conformal maps, we see in general that if z and w are inside the same box, then for all $t \geq 0$:

$$|h_t(z_1) - h_t(z_2)| \leq 2|\Im(z_1)h_t'(z_1)| \exp(4d_{\mathbb{H},hyp}(z_1,z_2))$$

$$2|\Im(z_1)h_t'(z_1)| \exp(4d_{\mathbb{H},hyp}(z_1,z_2)) \leq \Im(z_1)^{1-\beta}\phi(n) \leq \frac{\phi(n)}{\sqrt{n}^{1-\beta}} \to 0$$

  as $n \to \infty$.

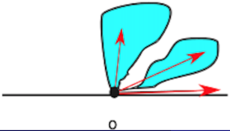
- We define for all fixed $t > 0$, the solution as $h_t(0+) = \lim_{n\to\infty} h_t(z_n)$. Note that the result holds for $\kappa \neq 8$. However, the convex cover of the sequence of boxes $A_{n,c,\phi}$, does not cover all possible directions to the origin in $\mathbb{H}$.

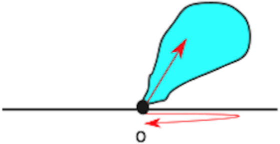
Applications to SLE

Proposition (Part 2)

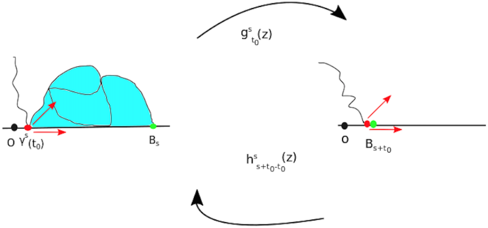
Let $\kappa > 4$ and let $\gamma^{(s)}(t_0) \in \mathbb{R}$ be a point of intersection of the SLE trace with the real line. Then, there are a.s. two solutions for the backward Loewner differential equation starting from $\gamma^{(s)}(t_0) \in \mathbb{R}$. One type that evolves in $\mathbb{H}$ and another type that evolves on $\mathbb{R}$.

- *For general hulls, for fixed $t > 0$, a.s. there can not be two or more gaps between the multiple solutions emerging from the origin up to time t for $\kappa \in (4, 8)$. Also, for $\kappa \in (4, 8)$ there can not be gap for all $t \in \mathbb{R}$, between the solutions that evolve in $\mathbb{H}$ and the ones that evolve along $\mathbb{R}$, a.s..*





The non-uniqueness for $\kappa > 4$



- For each $s > 0$, let us define $B_t^{(s)} := B_{t+s} - B_s$. Let $h^{(s)}(t)$ be the backward Loewner flow driven by the Brownian motion $\sqrt{\kappa}B_t^{(s)}$.
- For $\kappa > 4$, it can be shown that with probability 1, there exists $s > 0$ such that $\gamma^{(s)}(0,\infty) \cap [0, B_s] \neq \emptyset$.
- If we start the backward Loewner flow from B_{s+t_0} with driver $B_{s+t_0-\zeta}$, then we can make sense of solutions in $\mathbb{H}$ and along $\mathbb{R}$.

Continuity of the welding homeomorphism

The conformal welding homeomorphism is a homeomorphism of intervals of the real line given by the following rule: two points $x > 0$ and $y < 0$ are to be identified if they hit zero simultaneously under the backward Loewner differential equation.

Proposition

The welding homeomorphism induced by the backward Loewner differential equation on the real line when driven by $\sqrt{\kappa}B_t$ is a.s. (sequentially) continuous in the parameter κ, for $\kappa \in [0, 4]$.

- We prove that these homeomorphisms, as functions from $\mathbb{R}$ to $\mathbb{R}$ depending on the parameters κ_i, $i \in I$, are point-wise convergent a.s..
- We focus on $\kappa \in [0, 4]$, since in this regime there is link between the traces and the welding homeomorphism.

Elements of the proof

The Lamperti relation gives that $T^{\kappa}_{(0)}(x_0) = x_0^2 \int_0^{\infty} exp(2\tilde{B}_s + 2\mu(\kappa)s)ds$, with law

$$x_0^2 \int_0^{\infty} exp2(\tilde{B}_s + \mu(\kappa)s)ds \overset{(d)}{=} \frac{x_0^2}{2Z_{-\mu(\kappa)}},$$

where $Z_{-\mu(\kappa)}$ is a Gamma random variable with index $\mu(\kappa)$.

Lemma

For almost every Brownian path, no two points on the same side of the singularity can give rise to solutions to the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ that will hit the origin at the same time.

[Sketch of the proof] Let us consider $0 < x_0 < y_0$. For a.e. Brownian path, we have that

$$\frac{d}{dt}(y(t) - x(t)) = 2\frac{y(t) - x(t)}{(y(t) - \sqrt{\kappa}B_t)(x(t) - \sqrt{\kappa}B_t)}.$$

Elements of the proof

Lemma

Let us consider $X_t^{\kappa_i}(x_0)$ a collection of Bessel processes started from fixed $x_0 > 0$ coupled by driving them with the same Brownian paths.
Let $(\kappa_i)_{i \in I}$ be a strictly increasing sequence of values of $\kappa \in \mathbb{R}_+$. Then for all starting points $x_0 > 0$, $T_{(0)}^{\kappa_i}(x_0) \leq T_{(0)}^{\kappa_j}(x_0)$ for $i \leq j$, for almost every Brownian path.

- Considering the Laplace transforms of these times, we have that

$$\mathcal{L}(T_{(0)}^{\kappa_0 -}(x_0)) \leq \mathcal{L}(T_{(0)}^{\kappa_0}(x_0)) \leq \mathcal{L}(T_{(0)}^{\kappa_0 +}(x_0)).$$

- Thus, using the convergence of the laws in $\mu(\kappa)$, we obtain that

$$\mathcal{L}(T_{(0)}^{\kappa_0 +}(x_0)) = \lim_{\kappa_i \to \kappa_0} \mathcal{L}(T_{(0)}^{\kappa_i}(x_0)) = \mathcal{L}(T_{(0)}^{\kappa_0}(x_0)).$$

- We have that a.s. $[T_{(0)}^{\kappa_0 +}(x_0) - T_{(0)}^{\kappa_0}(x_0)](\omega) \geq 0$. Thus, we have a.s.

$$T_{(0)}^{\kappa_0 +}(x_0) = T_{(0)}^{\kappa_0}(x_0) = T_{(0)}^{\kappa_0 -}(x_0).$$

Elements of the proof

- We use the Lemma to argue that for the fixed value κ_{i_1}, there is a.s. only one point at the left symmetric with respect to the singularity that will hit the origin in time $T_{(0)}^{\kappa_{i_1}}(x_0)$. We call this point $y_0^{\kappa_{i_1}}$.

- If we keep the starting point fixed and we change the parameters, we obtain that the hitting time $T_{(0)}^{\kappa_i}(x_0) \to T_{(0)}^{\kappa_0}(x_0)$. The same applies symmetrically with respect to the origin.

- We consider the Lemma in order to get that for the fixed value κ_0 the point $y_0^{\kappa_0}$ that hits simultaneously with $x_0 > 0$ (for $\kappa = \kappa_0$) should be the same with the limiting $y_0^{\lim_i \kappa_i}$, where $\kappa_i \to \kappa_0$.

Forward Bessel processes and the control of the nullsets when varying $\kappa \in \mathbb{R}, \kappa \neq 8$.

1)The absolute continuity of the laws of the Bessel processes with various parameters given by the following relation:

$$\frac{d\mathbb{P}_{x_0}^{\mu(\kappa_i)}}{d\mathbb{P}_{x_0}^{\nu(\kappa_i)}}|_{\mathcal{F}_t^{\nu}} = \left(\frac{X_t}{x_0}\right)^{\mu(\kappa_i)-\nu(\kappa_i)} exp\left(-\frac{\mu^2(\kappa_i)-\nu^2(\kappa_i)}{2}\int_0^2 \frac{ds}{X_s^2}\right),$$

for any starting point $x_0 > 0$, $\mathbb{P}_{x_0}^{\nu(\kappa_i)}$- a.s..

- We use the rigidity in the coupled picture to control the dependence on the starting point.

2)Another useful property: $\mathbb{P}$-a.s. continuity in $(t,x) \in (0,\infty) \times [0,\infty)$ of the Bessel flow for dimension $d > 1$.

Strategy of the proof

Theorem (Tran)

Suppose the driving function of the Loewner chain $g_t(z)$ is weakly Hölder $\frac{1}{2}$ with subpower function ϕ and suppose that $|f_t'(iy)| \leq c_0 y^{-\beta}$ is satisfied. Then, there exists a subpower function $\tilde{\phi}(n)$ that depends on ϕ, c_0 and β, such that for all $n \geq \frac{1}{y_0^2}$ and $t \in [0,1]$ we have that $|\gamma^n(t) - \gamma(t)| \leq \frac{\tilde{\phi}(n)}{n^{\frac{1}{2}\left(1-\sqrt{\frac{1+\beta}{2}}\right)}}$, where γ^n is the curve generated by the algorithm.

- The constants used in the algorithm in front of $\sqrt{t}$ terms are sampled from a normal distribution $\mathcal{N}(0, 1/\sqrt{t})$ that is non-atomic.
- One challenge is to control the nullset outside of which the estimate $|f_t'(iy)| \leq c_0 y^{-\beta}$ holds when varying $8 \neq \kappa \in \mathbb{R}_+$. We do this using the forward Bessel flow on the real line.

Strategy of the proof

- We assume that the nullset outside of which the estimate $|f_t'(iy)| \leq c_0 y^{-\beta(\kappa_i)}$ is satisfied is changing as we vary κ_i with $i \in \mathbb{R}$.
- For fixed i, when $|f_t'(iy)| \leq c_0 y^{-\beta(\kappa_i)}$ holds, we have a.s. the existence of the trace and also we have that a.s. the collection of real Bessel processes of dimension $d(\kappa_i)$ exist.
- The collection of Brownian motions outside of which the absolute continuity of the laws of these processes holds, depends on the starting point. To control this, we use the rigidity of the dynamics in the coupled picture.
- We obtain a contradiction with the absolutely continuous laws of the Bessel processes of different parameters.

We currently try to use the rigidity also in the backward flow case.

Thank you for your attention!