Convergence in High Probability of the Quantum Diffusion in a Random Band Matrix Model

Project supervised by Prof. Antti Knowles- ETH Zürich

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Overview

Introduction of the model

Preliminaries

Introduction of the model

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- Let $\mathbb{Z}^d$ be the infinite lattice with the Euclidean norm $|\cdot|_{\mathbb{Z}^d}$ and let M the number of points situated at distance at most W ($W \geq 2$) from the origin, i.e.

$$M = M(W) = |\{x \in \mathbb{Z}^d : 1 \leq |\cdot|_{\mathbb{Z}^d} \leq W\}\,.$$

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- For simplicity we consider throught the proof a d-dimensional finite periodic lattice $\Lambda_N \subset \mathbb{Z}^d$ ($d \geq 1$) of linear size N equipped with the Euclidean norm $|\cdot|_{\mathbb{Z}^d}$. Specifically, we take Λ_N to be a cube centered around the origin with side length N, i.e.

$$\Lambda_N := ([-N/2, N/2) \cap \mathbb{Z})^d .$$

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We define the *random band matrix* (H_{xy}) through

$$H_{xy} := \sqrt{S_{xy}} A_{xy} ,$$

where (A_{xy}) is Hermitian random matrix whose upper triangular entries $(A_{xy} : x \leq y)$ are independent random variables uniformly distributed on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$.

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- Note that the entries H_{xy} in the random band matrix H are indexed by x and y which are indices of points in Λ_N .

Introduction of the Model

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$$t = W^{d\kappa} T\,,$$
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- For $\kappa > 0$, we introduce the macroscopic time and space coordinates T and X, which are independent of W, and consider the microscopic time and space coordinates
$$\begin{aligned} t &= W^{d\kappa}T\,,\\ x &= W^{1+d\kappa/2}X\,. \end{aligned}$$
- Given $\phi \in C_b(\mathbb{R}^d)$, we define the main quantity that we investigate by
$$Y_{T,\kappa,W}(\phi) \equiv Y_T(\phi) := \sum_x P(W^{d\kappa}T,x)\phi\left(\frac{x}{W^{1+d\kappa/2}}\right)\,.$$

Known and new result

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Theorem (Ërdos L. , Knowles A. 2011)

Let $0 < \kappa < 1/3$ *be fixed. Then for any* $\phi \in C_b(\mathbb{R}^d)$ *and for any* $T_0 > 0$ *we have that*

$$\lim_{W \to \infty} \mathbb{E} Y_T(\phi) = \int_{\mathbb{R}^d} dX \, L(T, X)\phi(X) \, ,$$

uniformly in $N \geq W^{1+d/6}$ *and* $0 \leq T \leq T_0$ *.*
Here

$$L(T, X) := \int_0^1 d\lambda \frac{4}{\pi} \frac{\lambda^2}{\sqrt{1-\lambda^2}} G(\lambda T, X) \, ,$$

and G *is the heat kernel*

$$G(T, X) := \left(\frac{d+2}{2\pi T} \right)^{d/2} e^{-\frac{d+2}{2T}|X|^2} \, .$$

Known and new result

Theorem (Knowles A., M. 2015)

Fix $T_0 > 0$ *and* κ *such that* $0 < \kappa < 1/3$. *Choose a real number* β *satisfying* $0 < \beta < 2/3 - 2\kappa$. *Then there exists* $C \geq 0$ *and* $W_0 \geq 0$ *depending only on* T_0, κ *and* β *such that for all* $T \in [0, T_0]$, $W \geq W_0$ *and* $N \geq W^{1+\frac{d}{6}}$ *we have*

$$Var(Y_T(\phi)) \leq \frac{C||\phi||_\infty^2}{W^{d\beta}}.$$

Sketch of the Proof

Expanding in non-backtracking powers

Let us introduce

$$H^{(1)} = H\,,$$

$$H^{(n)}_{x_0,x_n} := \sum_{x_1,\ldots,x_{n-1}} \left(\prod_{i=0}^{n-2} \mathbf{1}(x_i \neq x_{i+2}) \right) H_{x_0 x_1}, \ldots, H_{x_{n-1} x_n} \quad (n \geq 2)\,.$$

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Let U_k be the k-th Cebyshev polynomial of the second kind and let

$$\alpha_k(t) := \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-\zeta^2} e^{-it\zeta} U_k(\zeta) d\zeta .$$

We define the quantity $a_m(t) = \sum_{k \geq 0} \frac{\alpha_{m+2k}(t)}{(M-1)^k}$. Then we have that

$$e^{-itH/2} = \sum_{m \geq 0} a_m(t) H^{(m)} .$$

Expansion in non-backtracking powers

Expansion in non-backtracking powers

Plugging in the definition of $Y_T(\phi)$ we have

$$Var(Y_T(\phi)) = \sum_{y_1,y_2} \phi\left(\frac{y_1}{W^{1+d\kappa/2}}\right) \phi\left(\frac{y_2}{W^{1+d\kappa/2}}\right) \langle P(t,y_1); P(t,y_2)\rangle$$

$$\leq ||\phi||_\infty^2 \sum_{y_1} \sum_{y_2} |\langle P(t,y_1); P(t,y_2)\rangle| \,.$$

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Moreover,

$$\langle P(t,y_1); P(t,y_2)\rangle =$$

$$= \sum_{n_{11},n_{12}\geq 0} \sum_{n_{21},n_{22}\geq 0} a_{n_{11}}(t)\overline{a_{n_{12}}(t)}a_{n_{21}}(t)\overline{a_{n_{22}}(t)}\langle H_{0y_1}^{(n_{11})} H_{y_10}^{(n_{12})}; H_{0y_2}^{(n_{21})} H_{y_20}^{(n_{22})}\rangle \,.$$

Graphical Representation

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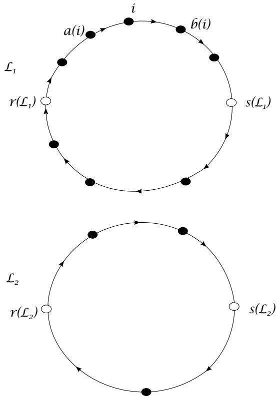

Figure: The graphical representation of a path of vertices.

- Each vertex $i \in V(\mathcal{L})$ carries a *label* $x_i \in \Lambda_N$.

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- For each configuration of labels $\mathbf{x}$ we assign a *lumping* $\Gamma = \Gamma(\mathbf{x})$ of the set of edges $E(\mathcal{L})$. The lumping $\Gamma = \Gamma(\mathbf{x})$ associated with the labels $\mathbf{x}$ is given by the equivalence relation

$$e \sim e' \Leftrightarrow \{x_{a(e)}, x_{b(e)}\} = \{x_{a(e')}, x_{b(e')}\} .$$

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- Using the graph $\mathcal{L}$ we may now write the covariance as

$$\langle H_{0y_1}^{(n_{11})} H_{y_1 0}^{(n_{12})}; H_{0y_2}^{(n_{21})} H_{y_2 0}^{(n_{22})} \rangle = \sum_{\mathbf{x} \in \Lambda_N^{V(\mathcal{L})}} Q_{y_1, y_2}(\mathbf{x}) A(\mathbf{x}) ,$$

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  where

$$Q_{y_1,y_2}(\mathbf{x}) = \mathbf{1}(x_{r(\mathcal{L}_1)} = 0)\mathbf{1}(x_{r(\mathcal{L}_2)} = 0)\mathbf{1}(x_{s(\mathcal{L}_1)} = y_1)\mathbf{1}(x_{s(\mathcal{L}_2)} = y_2) \prod_{i \in V_b(\mathcal{L})} \mathbf{1}(x_{a(i)} \neq x_{b(i)}).$$

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and

$$A(\mathbf{x}) = \mathbb{E} \prod_{e \in E(\mathcal{L})} H_{x_e} - \mathbb{E} \prod_{e \in E(\mathcal{L}_1)} H_{x_e} \mathbb{E} \prod_{e \in E(\mathcal{L}_2)} H_{x_e} .$$

Graphical Representation

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- Let $\mathfrak{P}_c(E(\mathcal{L}))$ be the set of connected even lumpings, i.e. the set of all lumpings Γ for which each lump $\gamma \in \Gamma$ has even size and there exists $\gamma \in \Gamma$ such that $\gamma \cap E(\mathcal{L}_k) \neq \emptyset$, for $k \in \{1, 2\}$.

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Lemma

We have that

$$\langle H_{0y_1}^{(n_{11})} H_{y_1 0}^{(n_{12})}; H_{0y_2}^{(n_{21})} H_{y_2 0}^{(n_{22})} \rangle = \sum_{\Gamma \in \mathfrak{P}_c(E(\mathcal{L}))} V_{y_1, y_2}(\Gamma),$$

where $V_{y_1,y_2}(\Gamma) = \sum_{\mathbf{x}} \mathbf{1}(\Gamma(\mathbf{x}) = \Gamma) Q_{y_1,y_2}(\mathbf{x}) A(\mathbf{x})$.

Graphical Representation

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- We define $\mathfrak{M}_c$ the set of all connected pairings

$$\bigsqcup_{n_{11},n_{12},n_{21},n_{22}} \{\Pi \in \mathfrak{P}_c(E(\mathcal{L}(n_{11}, n_{12}, n_{21}, n_{22}))) \ : \ |\pi| = 2\,, \ \ \forall \pi \in \Pi\}\,.$$

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- We call the lumps $\pi \in \Pi$ of a pairing Π *bridges*.

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$$J_{\{e,e'\}}(\mathbf{x}) \;=\; \mathbf{1}(x_{a(e)} = x_{b(e')})\mathbf{1}(x_{a'(e)} = x_{b(e)})\,.$$

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Lemma

We have

$$|\langle P(t,y_1); P(t,y_2)\rangle| \\ \leq \sum_{\Pi\in\mathfrak{M}_c} |a_{n_{11}(\Pi)}(t)\overline{a_{n_{12}(\Pi)}(t)}a_{n_{21}(\Pi)}(t)\overline{a_{n_{22}(\Pi)}(t)}| \\ \sum_{\mathbf{x}} Q_{y_1,y_2}(\mathbf{x}) \prod_{\{e,e'\}\in\Pi} S_{x_e} \prod_{\{e,e'\}\in\Pi} J_{\{e,e'\}}(\mathbf{x})\,.$$

Collapsing of parallel bridges

Collapsing of parallel bridges

- To each $\Pi \in \mathfrak{M}_c$ we associate a couple (Σ, l_Σ), where $\Sigma \in \mathfrak{M}_c$ has no parallel bridges and $l_\Sigma := (l_\sigma)_{\sigma \in \Sigma} \in \mathbb{N}^\Sigma$. The integer l_σ denotes the number of parallel bridges of Π that were collapsed into the bridge σ of Σ. Inverting the procedure we obtain a bijection $\Pi \longleftrightarrow (\Sigma, l_\Sigma)$.

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- We further define the set of admissible skeletons as

$$\mathfrak{G} \;=\; \{S(\Pi) : \Pi \in \mathfrak{M}_c\}\,.$$

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- We further define $|l_\Sigma| = \sum_{\sigma \in \Sigma} l_\sigma$ for $\Sigma \in \mathfrak{G}$.

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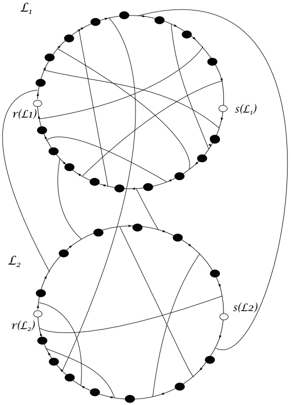

Figure: Graphical representation of the skeleton for a given configuration

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We have that

$$\sum_{y_1}\sum_{y_2}\langle P(t,y_1);P(t,y_2)\rangle \le$$
$$\sum_{\Sigma\in\mathfrak{G}}\sum_{l_\Sigma}|a_{n_{11}(\Sigma,l_\Sigma)}(t)\overline{a_{n_{12}(\Sigma,l_\Sigma)}(t)}a_{n_{21}(\Sigma,l_\Sigma)}(t)\overline{a_{n_{22}(\Sigma,l_\Sigma)}(t)}|R(\Sigma)\,,$$

where

$$R(\Sigma)=\sum_{\mathbf{x}\in\Lambda_N^{V(\Sigma)}}\mathbf{1}(x_{r(\mathcal{L}_1(\Sigma))}=0)\mathbf{1}(x_{r(\mathcal{L}_2(\Sigma))}=0)$$
$$\prod_{\{e,e'\}\in\Sigma}\left(S^{l_{\{e,e'\}}}\right)_{x_e}\prod_{\sigma\in\Sigma}J_\sigma(\mathbf{x})\,.$$

Orbits of vertices

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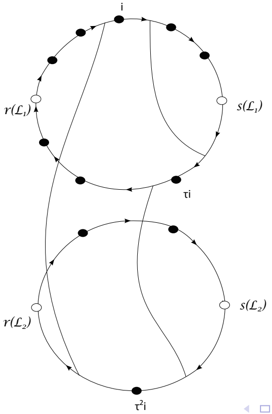
- For fixed $\Sigma \in \mathfrak{G}$ we define $\tau : V(\Sigma) \to V(\Sigma)$ as follows. Let $i \in V(\Sigma)$ and let e be the unique edge such that $\{\{i, b(i)\}, e\} \in \Sigma$. Then, for any vertex i of $\Sigma \in \mathfrak{G}$ we define $\tau i = b(e)$. We denote the orbit of the vertex $i \in \Sigma$ by $[i] := \{\tau^n i : n \in \mathbb{N}\}$



Orbits of vertices

- Let $Z(\Sigma) := \{[i] : i \in V(\Sigma)\}$ be the set of orbits of Σ and $|\Sigma|$ be the number of bridges of the skeleton Σ and let $L(\Sigma) = |Z^*(\Sigma)|$ with $Z^*(\Sigma) := Z(\Sigma) \setminus \{[r(\mathcal{L}_1)], [r(\mathcal{L}_2)]\}$.

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We have the inequality

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Lemma

Let $\Sigma \in \mathfrak{G}$ and $I_\Sigma \in \Lambda_N^{V(\Sigma)}$. We have that

$$R(\Sigma) \;\leq\; C\left(\frac{M}{M-1}\right)^{|I_\Sigma|} M^{-|\Sigma|/3+2/3}\,.$$

Truncation

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- We introduce a cut-off at $|l_\Sigma| < M^\mu$ for $\mu < 1/3$. We define

$$E^{\leq} = \sum_{\Sigma \in \mathfrak{G}} \sum_{|l_\Sigma| \leq M^\mu} |a_{n_{11}(\Sigma, l_\Sigma)}(t) \overline{a_{n_{12}(\Sigma, l_\Sigma)}(t)} a_{n_{21}(\Sigma, l_\Sigma)}(t) \overline{a_{n_{22}(\Sigma, l_\Sigma)}(t)}| R(\Sigma)$$

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Lemma

1. *For any time t and for any $n \in \mathbb{N}$ we have $|a_n(t)| \leq \frac{Ct^n}{n!}$, with C universal constant.*
2. *We have $\sum_{n \geq 0} |a_n(t)|^2 = 1 + O(M^{-1})$, uniformly in $t \in \mathbb{R}$.*

Main Lemma

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Lemma

For any $\Sigma \in \mathfrak{G}$ *with* $|\Sigma| \geq 3$ *we have*

$$\sum_{I_\Sigma} \mathbf{1}(|I_\Sigma| \ \leq \ M^\mu)|a_{n_{11}(\Sigma,I_\Sigma)}(t)\overline{a_{n_{12}(\Sigma,I_\Sigma)}(t)}a_{n_{21}(\Sigma,I_\Sigma)}(t)\overline{a_{n_{22}(\Sigma,I_\Sigma)}(t)}|$$

$$\leq \frac{CM^{\mu(|\Sigma|-2)}}{(|\Sigma|-3)!}.$$

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In order to finish the argument

1. We prove using Stirling approximation and some arguments involving geometric series that the remaining terms are tiny.

$$E^{>} = \sum_{\Sigma \in \mathfrak{G}} \sum_{|I_\Sigma| \geq M^\mu} |a_{n_{11}(\Sigma, I_\Sigma)}(t) \overline{a_{n_{12}(\Sigma, I_\Sigma)}(t)} a_{n_{21}(\Sigma, I_\Sigma)}(t) \overline{a_{n_{22}(\Sigma, I_\Sigma)}(t)}| R(\Sigma)$$

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2. We find bounds by direct computation for the cases $|\Sigma| = 0$, $|\Sigma| = 1$ and $|\Sigma| = 2$, and we conclude the proof.

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- We started with

$$Var(Y_T(\phi)) \;\leq\; ||\phi||_\infty^2 \sum_{y_1} \sum_{y_2} |\langle P(t,y_1); P(t,y_2)\rangle| \, .$$

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$$Var(Y_T(\phi)) \le ||\phi||_\infty^2 \sum_{y_1} \sum_{y_2} |\langle P(t,y_1); P(t,y_2)\rangle| \,.$$

- Via graphical and combinatorial arguments we arrived at

$$\sum_{y_1} \sum_{y_2} \langle P(t,y_1); P(t,y_2)\rangle \le$$

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- Truncating the expression in $|l_\Sigma|$ and summing the two terms under the specific conditions on the parameters imposed by the setting (i.e. $0 < \kappa \leq 1/3, .\ 0 < \beta < 2/3 - 2\kappa\,.$), we obtain a bound for the variance of the form

$$Var(Y_T(\phi)) \leq \frac{C||\phi||^2_\infty}{W^{d\beta}}\,.$$

Thank you for your attention!

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