Truncated Taylor approximation of Loewner dynamics

Supervised by Prof. Dmitry Belyaev and Prof. Terry Lyons

Vlad Margarint

Dept. of Mathematics, University of Oxford

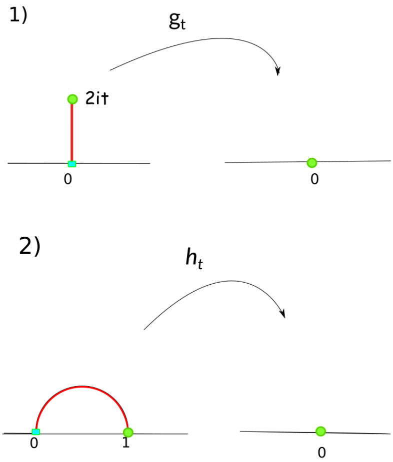
vlad.margarint@maths.ox.ac.uk

Berlin WIAS
August 2016

Overview

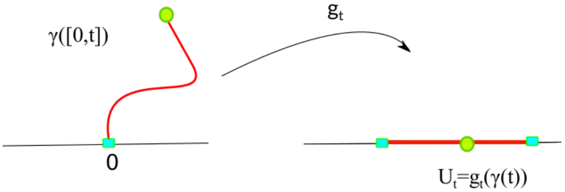
Conformal maps

- Examples of conformal maps from upper halfplane with a slit to the upper halfplane $\mathbb{H}$.



Conformal maps and the Loewner equation

- In general, for a non-self crossing curve $\gamma(t) : [0, \infty) \to \bar{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(\infty) = \infty$, we consider the simply connected domain $\mathbb{H} \setminus \gamma([0, t])$.



- Using the Riemann Mapping Theorem for the simply connected domain $\mathbb{H} \setminus \gamma([0, t])$, we have a three real parameter family of conformal maps $g_t : \mathbb{H} \setminus \gamma([0, t]) \to \mathbb{H}$.
- Loewner Equation encodes the dependence between the evolution of the maps g_t when the curve $\gamma([0, t])$ grows.

Description of the conformal maps

- Setting the behaviour of the mapping at ∞ as $g_t(\infty) = \infty$ and $g_t'(\infty) = 1$, we write the Laurent expansion at ∞ of g_t as

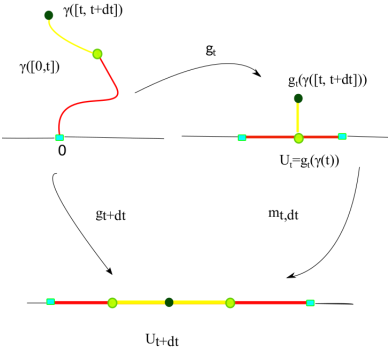
$$g_t(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$$

- We fix the third paramater by choosing $b_0 = 0$.
- The coefficient $b_1 = b_1(\gamma([0,t]))$ is called the *half-plane capacity* of $\gamma(t)$ and is proved to be an additive, continous and increasing function. Hence, by reparametrizing the curve $\gamma(t)$ such that $b_1(\gamma([0,t])) = 2t$, we obtain

$$g_t(z) = z + \frac{2t}{z} + \dots$$

Conformal maps and the Loewner equation

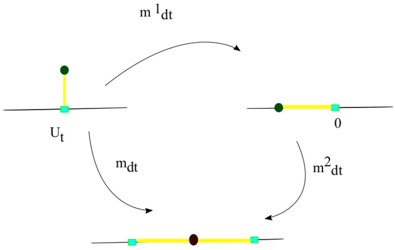
- Is there a way to use g_t to find g_{t+dt} ?



- In order to answer this question, we have to describe to find a way to describe the mapping $m_{t,dt} : \mathbb{H} \setminus g_t(\gamma[t, t+dt]) \to \mathbb{H}$.

The Loewner equation and the square root map

- The square root map that we investigated in the beginning gives the description of the 'infinitesimal mapping' $m_{t,dt}$.



- Heuristically, $m_{t,dt}(z) = U_{t+dt} + \sqrt{(z - U_t)^2 + 2dt} \approx z + \frac{2dt}{z - U_t}$.
- Furthermore, $g_{t+dt}(z) \approx g_t(z) + \frac{2dt}{g_t(z) - U_t}$.
- We obtain the Loewner Differential Equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z .$$

Loewner equation and random curves in the upper half-plane

- So far, we adopted the perspective that given the curve γ_t , the conformal maps g_t must satisfy

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z\,.$$

with $U_t = g(\gamma(t))$.

- From now on, we take the dual perspective. Given the driving function $U_t : [0, \infty) \to \mathbb{R}$, we determine g_t . Then, the maps g_t determine the curve $\gamma(t)$.
- To output random continous curves, U_t has to be a random continous driver. Moreover, the random driver U_t induces a law on the curves $\gamma(t)$.

Definition of SLE and dependence on κ

Definition

Let B_t be a standard real Brownian motion starting from 0. The chordal SLE(κ) is defined as the law on curves induced by the solution to the following ordinary differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}\,, \qquad g_0(z) = z\,.$$

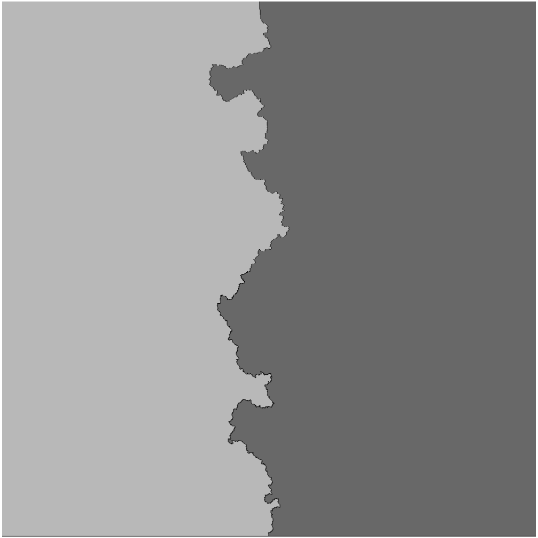


Figure: SLE(1): Credit Prof. Vincent Beffara

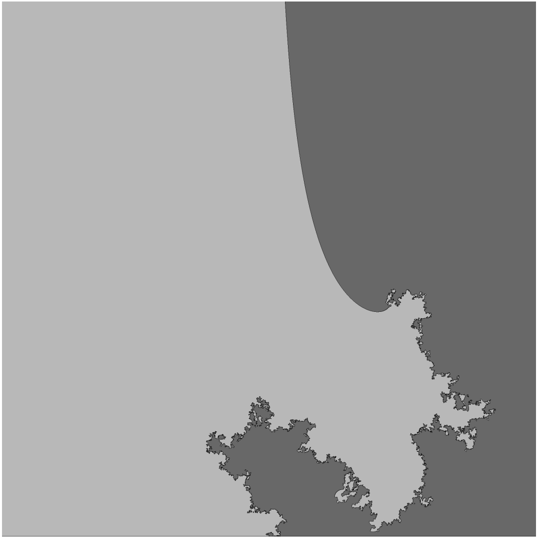


Figure: SLE(3.5): Credit Prof. Vincent Beffara

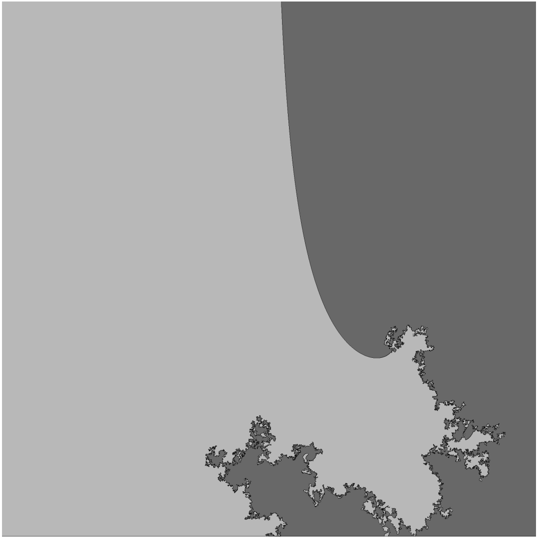


Figure: SLE(4.5): Credit Prof. Vincent Beffara

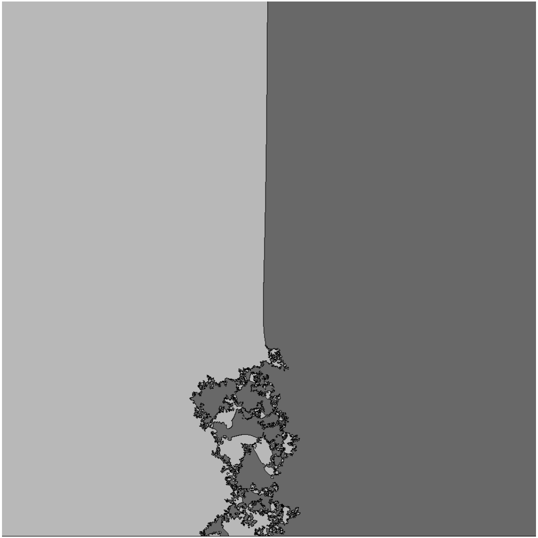
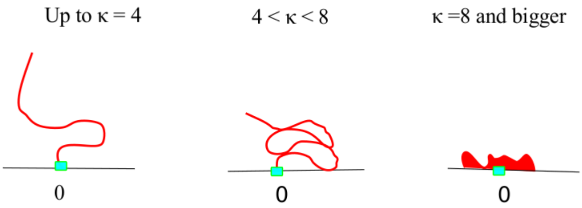


Figure: SLE(6): Credit Prof. Vincent Beffara

- It is proved that there are two phase transitions when κ varies between 0 and ∞ .



- The argument uses the phase transition of the Bessel process on the real line.
- In order to show this, consider the process $dZ_t = \frac{2dt}{\kappa Z_t} - dB_t$, where $Z_t := \frac{1}{\sqrt{\kappa}} g_t - B_t$.

SLE phase transitions

- When started with a real initial value, the process $dZ_t = \frac{2dt}{\kappa Z_t} - dB_t$ is a real valued Bessel process with parameter $a = \frac{2}{\kappa}$.
- If $\kappa \leq 4$, then with probability one , the hitting time of zero $T_x = \infty$ for all non-zero $x \in \mathbb{R}$.
- If $\kappa \geq 4$, then with probability one, the hitting time of zero $T_x < \infty$ for all non-zero $x \in \mathbb{R}$.
- If $4 < \kappa < 8$ and $x < y \in \mathbb{R}$, then $\mathbb{P}(T_x = T_y) > 0$.
- If $\kappa \geq 8$, then with probability one, $T_x < T_y$ for all reals $x < y$.

The Rough Paths perspective

- We consider the backward Loewner differential equation

$$\partial_t h_t(z) = \frac{-2}{h_t(z) - \sqrt{\kappa} B_t}, \qquad h_0(z) = z \,.$$

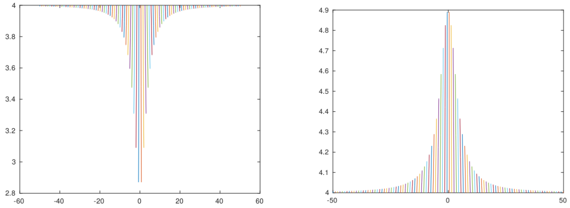


Figure: The images of a thin rectangle under the forward Loewner evolution (left) and backward Loewner evolution(right) for $\kappa = 0$.

- Finally, we obtain the following RDE in the upper half plane:

$$dz_t = \frac{-2}{z_t} dt - \sqrt{\kappa} dB_t \,.$$

The Lie bracket of the two vector fields and the uncorrelated diffusions

- We study an approximation to the solution of the RDE

$$dz_t = \frac{-2}{z_t}dt - \sqrt{\kappa}dB_t \, .$$

Remark

For $z = x + iy$ *, we have that* $[\frac{-2x}{x^2+y^2}\frac{\partial}{\partial_x} + \frac{2y}{x^2+y^2}\frac{\partial}{\partial_y}, \sqrt{\kappa}\frac{\partial}{\partial_x}] = \frac{-2\sqrt{\kappa}}{z^2}$.

Proposition

Let $\epsilon > 0$ *. At space scale* ϵ *and time scale* ϵ^2 *the increment of the horizontal Brownian motion* B_t *and the increment of the area process between* t *and* B_t *are uncorrelated. Moreover, they give the same order contribution in the approximation.*

The field of ellipses

- We consider the field of ellipses associated with this diffusion. Note that these ellipses should be shifted along the drift.
- At this specific scales the directions and lengths of the axes are computed explicitly in terms of the argument θ and the parameter κ.

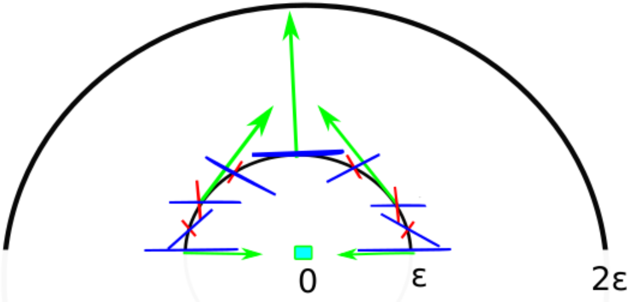


Figure: A schematic representation of the field of ellipses. The drift direction is represented in green.

Explicit dynamics and local truncation error up to the second level

Proposition

Fix $\epsilon > 0$. The truncated second level order Taylor approximation $\tilde{z}_t$ of the Loewner RDE started from $|z_0| = \epsilon$, at time $\epsilon^2 > 0$ is an explicit function of κ, z_0 and ϵ. Moreover, the local truncation error of the truncated Taylor approximation is $O(\epsilon)$.

- Important: the contribution of the second order approximation term

$$\int_0^{\epsilon^2} \frac{-2\sqrt{\kappa}}{Z_t^2} dA_t \text{ is } O(\epsilon)\,,$$

since $\frac{1}{|Z_0|^2} = \frac{1}{\epsilon^2}$ and $A_{\epsilon^2} = \frac{1}{2}\left(\int_0^{\epsilon^2} B_s ds - \int_0^t s dB_s\right)$ is $O(\epsilon^3)$.

Elements of the proof

- The diffusive part of the approximation is described by the ellipses given by
$$\left(T\begin{bmatrix}u\\ v\end{bmatrix}\right)^t\begin{bmatrix}A & B\\ C & D\end{bmatrix}T\begin{bmatrix}u\\ v\end{bmatrix}=1,$$
where
$$T=\begin{bmatrix}\sqrt{\kappa\epsilon^2} & -\mathrm{Re}\frac{1}{z^2}\sqrt{\frac{\epsilon^6}{3}\kappa}\\ 0 & -\mathrm{Im}\frac{1}{z^2}\sqrt{\frac{\epsilon^6}{3}\kappa}\end{bmatrix}.$$
- We obtain the explicit squares of semi-axis of the ellipses $a_{1,2}(\kappa,\theta,\epsilon)$ as inverses of the solutions to
$$\lambda^2-\lambda\left(\frac{1}{\kappa\epsilon^2}+\frac{3}{\kappa\epsilon^6\mathrm{Im}^2\frac{1}{z^2}}+\frac{ctg^2(-2\theta)}{\kappa\epsilon^2}\right)+\frac{3}{\kappa^2\epsilon^8\mathrm{Im}^2\frac{1}{z^2}}=0\,.$$

Future perspectives

- Compare the probability of crossing a sequence of centered annuli for the *Forward* Loewner evolution given by the Rough Paths approach with the one given by the typical Bessel process approach.
- Similarly, study the dynamics given by the Rough Paths approach on the boundary.
- Study in polar coordinates $arg(z_t)$ using the logarithm mapping.

Thank you for your attention!

References

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