# (*)Convergence in Probability of the Quantum diffusion in a Random Band Matrix model

# (**) Proof of the Weak Local Law for Wigner Matrices using ("only") Resolvent Expansions

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## Definitions and first result

- (*)Random band matrices are the natural objects that interpolate in between the Wigner matrices and Random Schrödinger operators. We consider Hermitian random band matrices $H$ in $d \geqslant 1$ dimensions. The matrix elements $H_{xy}$, indexed by $x, y \in \Lambda \subset \mathbb{Z}^d$, are independent, uniformly distributed random variables if $|x-y|$ is less than the band width $W$, and zero otherwise. We update the previous results of the convergence of quantum diffusion in a random band matrix model from convergence of the expectation to convergence in high probability. This new approach is similar with the one in [1], where it was proved that the quantum dynamics of the $d$-dimensional band matrix is given by a superposition of heat kernels up to time scales $t \ll W^{d/3}$. Note that diffusion is expected to hold for $t \sim W^2$ for $d=1$ and up to any time for $d \geqslant 3$ when the thermodynamic limit is taken. The threshold exponent $d/3$ is due to technical estimates on certain Feynman graphs. In this new approach we use double-rooted Feynman graphs to estimate the variance of the quantum diffusion.

Our main quantity is
$$P(t,x) = |(e^{-itH/2})_{0x}|^2 .$$
The function $P(t,x)$ describes the quantum transition probability of a particle starting at the origin 0 and ending in position $x$ after time $t$.

We define the random variable that we are going to investigate by
$$Y_{T,\kappa,W}(\phi) \equiv Y_T(\phi) := \sum_x P(W^{d\kappa}T, x)\phi\left(\frac{x}{W^{1+d\kappa/2}}\right),$$
where $\phi \in C_b(\mathbb{R}^d)$ is a test function in $\mathbb{R}^d$.

Our main result gives an estimate for the variance of the random variable $Y_T(\phi)$ up to time scales $t = O(W^{d\kappa})$ if $\kappa < 1/3$. Hence, as a Corollary we obtain the convergence in high probability of the quantum diffusion in this model.

**Theorem 0.1.** *Fix $T_0 > 0$ and $\kappa$ such that $0 < \kappa < 1/3$. Choose a real number $\beta$ satisfying $0 < \beta < 2/3 - 2\kappa$. Then there exists $C \geqslant 0$ and $W_0 \geqslant 0$ depending only on $T_0$, $\kappa$ and $\beta$ such that for all $T \in [0, T_0]$, $W \geqslant W_0$ and $N \geqslant W^{1+\frac{d}{6}}$ we have*
$$\mathrm{Var}(Y_T(\phi)) \leqslant \frac{C||\phi||_\infty^2}{W^{d\beta}} .$$

## Defintions and second result

- (**) We give a new proof of the Local Semicircle Law for the Wigner Ensemble by using intensively the algebraic structure of resolvent expansions. We combine this with concentration of measure results and high probability bounds. The conclusion is obtained using a bootstrapping argument that provides information about the change of the bounds from large to small scales. To simplify the notation in our Lemmas, we omit the $z$-dependence of $G(z)$.

Let $H$ be a Wigner Matrix, and let $\tilde{H} = H + \Delta$ be a perturbation of it. By iterating the resolvent identity $\tilde{G} = G - G\Delta\tilde{G}$, we get the resolvent expansion
$$\tilde{G}(z) = \sum_{i=0}^{N-1} G(z)(-\Delta G(z))^i + \tilde{G}(z)(-\Delta G(z))^N .$$
We say that $X$ is stochastically dominated by $Y$ (and we use the notation $X \prec Y$), uniformly in $u$, if for all $\varepsilon > 0$ and large $D > 0$ we have
$$\sup_{u \in U^{(N)}} \mathbb{P}\left[X^{(N)}(u) > N^{\varepsilon} Y^{(N)}(u)\right] \leqslant N^{-D}$$
for large $N \geqslant N_0(\varepsilon, D)$.

For fixed $\gamma \geqslant 0$ we define the spectral domain
$$\mathbf{S} \equiv \mathbf{S}_N(\gamma) := \{z = E + i\eta : -N \leqslant E \leqslant N,\ N^{-1+\gamma} \leqslant \eta \leqslant N\} .$$

**Theorem 0.2** (Local Semicircle Law for Wigner Matrices)**.** *Let $H$ be a $N \times N$ Wigner Matrix and let $\psi(z) := \frac{1}{\sqrt{N\eta}}$ be a deterministic error parameter. If $m(z)$ is the Stieltjes transform of the Semicircle Law $\rho(dx) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}dx$, then we have that*
$$\max_{i \in [\![1,\dots,N]\!]} |G_{ii}(z) - m(z)| \prec F_z(\psi(z)) ,$$
$$\max_{i \neq j} |G_{ij}(z)| \prec \psi(z) ,$$
*uniformly for all $z \in \mathbb{C}_+$ such that $\eta \geqslant N^{-1+\gamma}$, where* $F_z(r) := \left[\left(1 + \frac{1}{\sqrt{|z^2-4|}}\right) r\right] \wedge \sqrt{r}$.

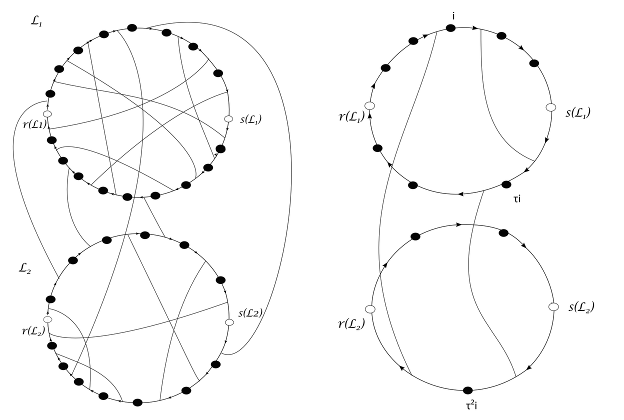
## Sketch of the proof of the first result

- We have the following bound for the main random variable.

$$\mathrm{Var}(Y_T(\phi)) = \langle Y_T(\phi); Y_T(\phi)\rangle = \sum_{y_1, y_2} \phi\left(\frac{y_1}{W^{1+d\kappa/2}}\right)\phi\left(\frac{y_2}{W^{1+d\kappa/2}}\right)\langle P(t,y_1); P(t,y_2)\rangle$$
$$\leqslant ||\phi||_\infty^2 \sum_{y_1}\sum_{y_2} |\langle P(t,y_1); P(t,y_2)\rangle| .$$

$$\langle P(t,y_1); P(t,y_2)\rangle = \sum_{n_{11}, n_{12} \geqslant 0}\ \sum_{n_{21}, n_{22} \geqslant 0} a_{n_{11}}(t)\overline{a_{n_{12}}(t)}a_{n_{21}}(t)\overline{a_{n_{22}}(t)}\langle H_{0y_1}^{(n_{11})} H_{y_10}^{(n_{12})}; H_{0y_2}^{(n_{21})} H_{y_20}^{(n_{22})}\rangle .$$

- In the left image is the skeleton $\Sigma = S(\Pi)$ of the pairing $\Pi$ after collapsing all the parallel bridges. In the right image is represented the orbit of the vertex $i$.



- To each pairing $\Pi$ we associate a couple $(\Sigma, l_\Sigma)$, where $\Sigma$ has no parallel bridges and $l_\Sigma := (l_\sigma)_{\sigma \in \Sigma} \in \mathbb{N}^\Sigma$. The integer $l_\sigma$ denotes the number of parallel bridges of $\Pi$ that were collapsed into the bridge $\sigma$ of $\Sigma$. In this manner we construct the set $\mathfrak{G}$ of admissible skeletons. If $M$ is the number of points on the lattice situated at distance at most the band width $W$ from the origin and if we denote by $|\Sigma|$ the number of bridges of the skeleton $\Sigma$, then we have that

$$\sum_{y_1}\sum_{y_2}\langle P(t,y_1); P(t,y_2)\rangle \leqslant C\sum_{\Sigma \in \mathfrak{G}}\sum_{l_\Sigma} |a_{n_{11}(\Sigma,l_\Sigma)}(t)\overline{a_{n_{12}(\Sigma,l_\Sigma)}(t)}a_{n_{21}(\Sigma,l_\Sigma)}(t)\overline{a_{n_{22}(\Sigma,l_\Sigma)}(t)}| \left(\frac{M}{M-1}\right)^{|l_\Sigma|} M^{-|\Sigma|/3+2/3} . \qquad (1)$$

- We introduce a cut-off in the summation in (1) at $|l_\Sigma| \leqslant M^\mu$ for $\mu < 1/3$. The main step of the proof of the final result is the following estimate for $\Sigma \in \mathfrak{G}$ with $|\Sigma| \geqslant 3$ (the cases $|\Sigma| = 1$ and $|\Sigma| = 2$ can be proved directly)

$$\sum_{l_\Sigma} \mathbf{1}(|l_\Sigma| \leqslant M^\mu)|a_{n_{11}(\Sigma,l_\Sigma)}(t)\overline{a_{n_{12}(\Sigma,l_\Sigma)}(t)}a_{n_{21}(\Sigma,l_\Sigma)}(t)\overline{a_{n_{22}(\Sigma,l_\Sigma)}(t)}| \leqslant \frac{CM^{\mu(|\Sigma|-2)}}{(|\Sigma|-3)!} .$$

## Sketch of the proof of the second result

[Concentration of measure Lemma]

Let $z \in \mathbf{S}$. If $|G_{kl}| \prec N^\delta$, then
$$|G_{kl} - \mathbb{E}_1 G_{kl}| \prec \frac{N^{3\delta/2}}{\sqrt{N\eta}}, \quad \text{for all } k, l \in [\![1, N]\!] .$$

⇒

[Bound on the average Lemma]

Let $s := \frac{1}{N}\sum_{j=1}^{N} G_{jj}$.

If $|G_{kl}| \prec N^\delta$ for $k, l \in [\![1, N]\!]$, it follows that for all $z \in \mathbf{S}$ we have
$$1 + zs + s^2 = O_\prec\left(\frac{(1+|z|)N^{5\delta}}{\sqrt{N\eta}}\right) .$$

⇒

[Bound on scale $\eta$ Lemma]

Let $\Gamma(z) := \max_{k,l}|G_{kl}(z)| \vee 1$ and let $\Gamma^*(z) := \sup_{\eta' \geqslant \eta}\Gamma(E + i\eta')$.

If for $z \in \mathbf{S}$ we have $\Gamma^*(z) \prec N^\delta$, it follows that
$$\max_{i \in [\![1,N]\!]} |m - G_{ii}| \prec F\left(\frac{N^{5\delta}}{\sqrt{N\eta}}\right) N^\delta ,$$
$$\max_{i \neq j \in [\![1,N]\!]} |G_{ij}| \prec \frac{N^{5\delta/2}}{\sqrt{N\eta}} .$$

⇒

[Bootstrap argument to obtain bounds at lower scales]

- For any $M > 1$ and $z \in \mathbb{C}_+$, we have
$$\Gamma(E + i\eta/M) \leqslant M\Gamma(E + i\eta) .$$
- Let $K := \max\{k \in \mathbb{N} : N/N^{k\delta} \geqslant N^{-1+\gamma}\}$. For $k \in [\![0, K]\!]$ let $z_k := E + i\eta_k$, where $\eta_k = \frac{N}{N^{k\delta}}$. Then we have that $\Gamma^*(z_k) \prec 1$.

## References

[1] (*) Knowles A. Erdös L. Quantum Diffusion and Eigenfunction delocalization in a Random Band Matrix Model. *Comm. Math. Phys.*, (303):509–554, 2011.

[2] (**) Knowles A. Benaych-Georges F. Lecture notes on the local semicircle law for Wigner matrices. *Panoramas et Synthses*, 2016.

[3] (**) Bauerschmidt R. Knowles A. Yau H.-T. Local Semicircle Law for Random Regular Graphs. *Preprint*, 2015.