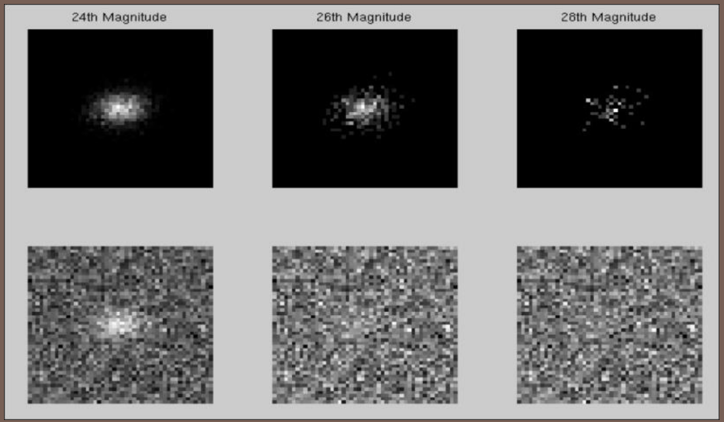
IMAGE ANALYSIS USING SHAPELETS FORMALISM AND COMPRESSIVE SENSING



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The problem & the context

1)Trying to recover images , in particular galaxy images made by Hubble Space Telescope.

2)The concept of recovery.

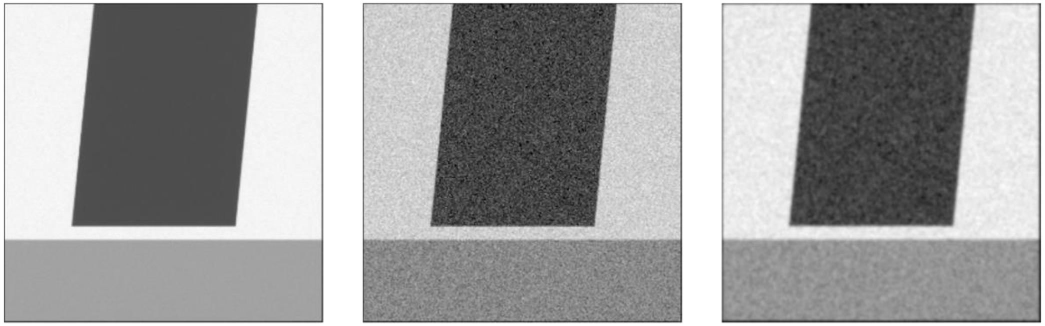
2.1)Noise;

$$Y=X+W;$$

Y=observed image;

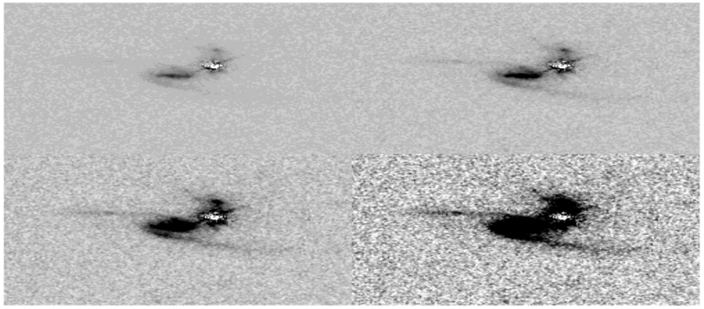
X=real image;

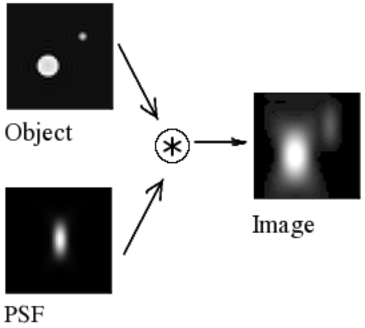
W=noise;



Low noise (Exp : 0.3s) High noise (Exp : 0.003s) De-noised

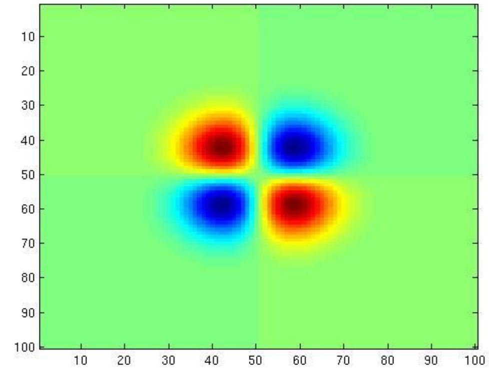
2.2)Point spread function;





1)Shapelets formalism

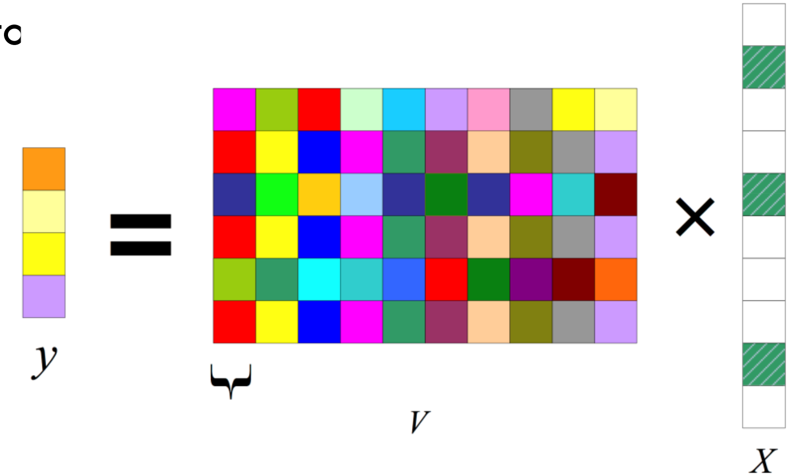
- Orthonormal basis of a vectorial space;
- -Used for representing the galaxies.



2)Compressive sensing

-

New efficient method of image analysingthat is trying to reconstruct a N-length signal by using only M<N measurements.



Compressive sensing method

The signal x in time domain can pe written in other basis of the vectorial space

$$\mathrm{x} = \sum_{i=1}^{N} s_i \, \psi_i \qquad \text{or} \qquad \mathrm{x} = \psi \mathrm{s}$$

The camera usually extracts all the coefficients with this method and keeps only K

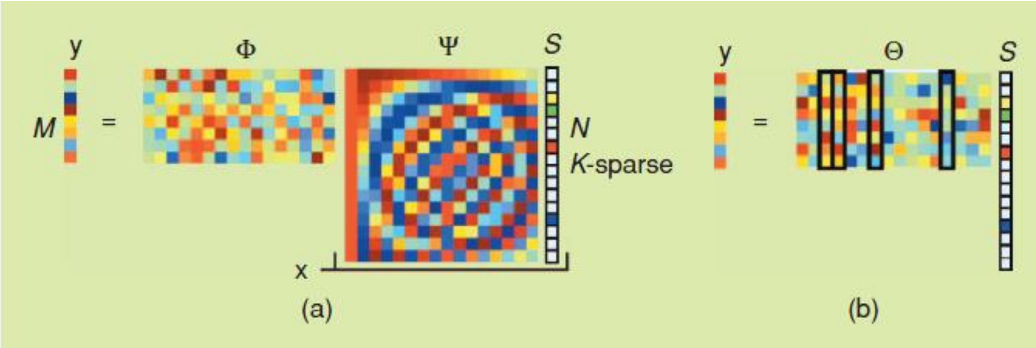
$$\mathbf{s} = \Psi^T \mathbf{x};$$

In this new method you will take m vectors with specific conditions $\{\phi_j\}_{j=1}^{M}$ and form a matrix

$$y_j = \langle \mathrm{x}, \phi_j \rangle .$$

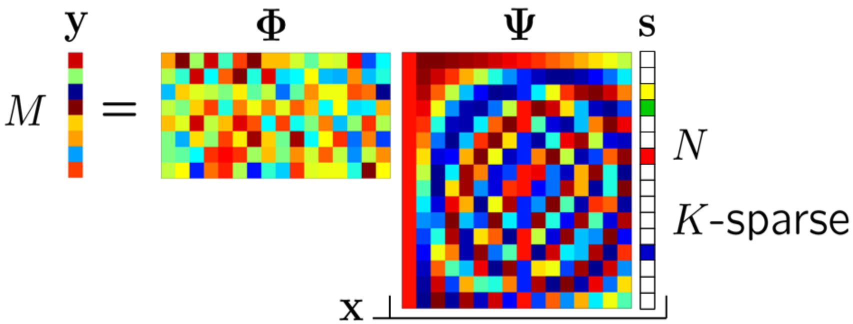
Finding x sparse signal from y the measurement vector

$$\mathbf{y} = \Phi \mathbf{x} = \Phi \Psi \mathbf{s} = \Theta \mathbf{s}$$



$$y = Ax + z,$$

Compressive sensing method



Subjected to min || ||1

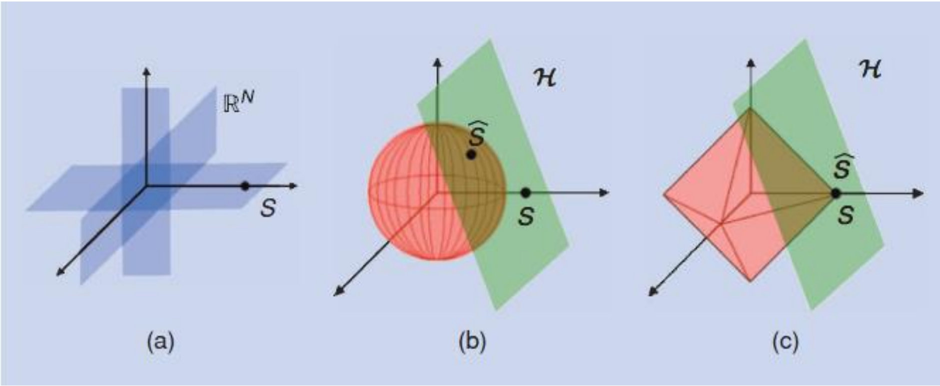


Fig.28

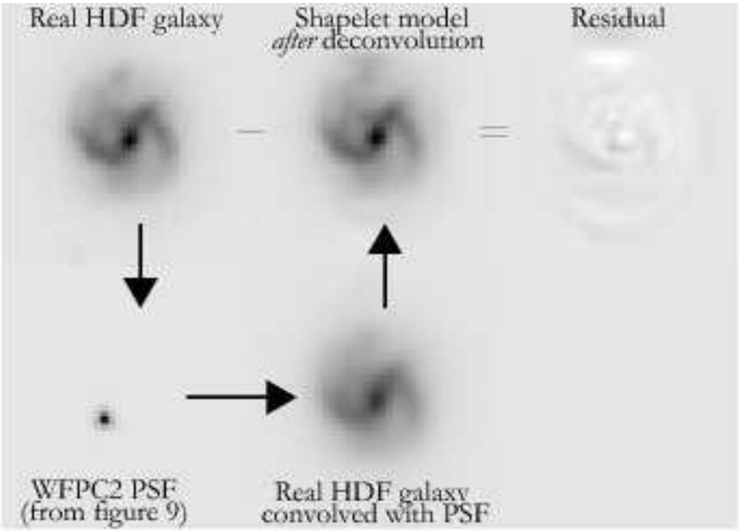


Fig.29

My work since the start of the project:

Part (1) The background of the theory:

--Hermite polynomials , Hermite Polar polynomials and Laguerre polynomials;

Part (2)

--Cartesian and Polar Shapelets for 1D and for 2D .

--Creating images with the help of the Shapelets.

1)Cartesian system: Hermite polynomials

The Hermite polynomials unnormalised recurrence relation:

$$H_P(x,n) = 2xH_P(n-1,x) - 2(n-1)H_P(n-2,x)$$

The Hermite polynomials normalised recurrence relation:

$$\tilde{H}_P(n,x) = x\sqrt{2/n}\tilde{H}_P(n-1,x) - \sqrt{n-1)/n}\tilde{H}_P(n-2,x); (2)$$

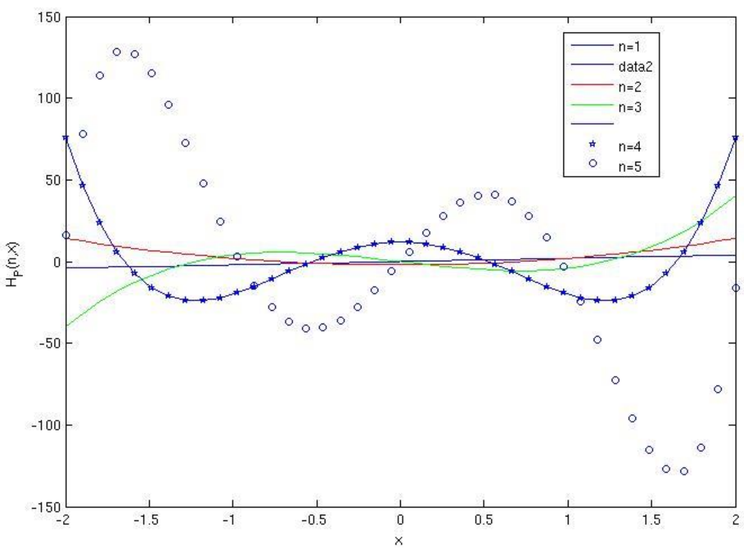


Fig.1

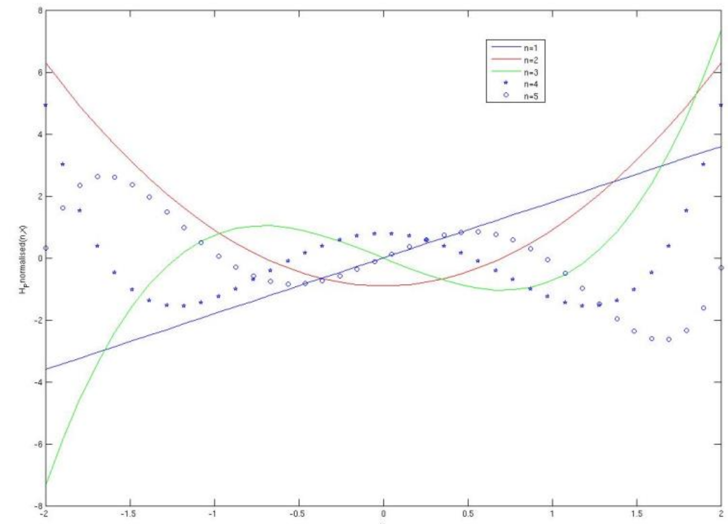


Fig.2

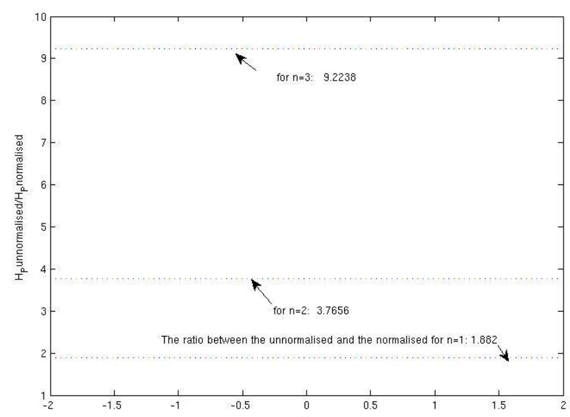


Fig.3

1)Cartesian system: Shapelets for 1D

- Shapelets for 1D that are obtained using the Hermite normalised polynomials weighted by an exponential function .

$$\phi_n(x) = Shpl(n,x) = \tilde{H}p(n,x) * exp((-x^2)/2)$$

$$\int_{-\infty}^{\infty} dx\ \phi_n(x)\phi_m(x) = \delta_{nm},$$

- Observation: They form an orthonormal basis.

Ortotest results:

- ortotest(2,2)
- ans =1.0000
- >> ortotest(1,0)
- ans =-3.6791e-18

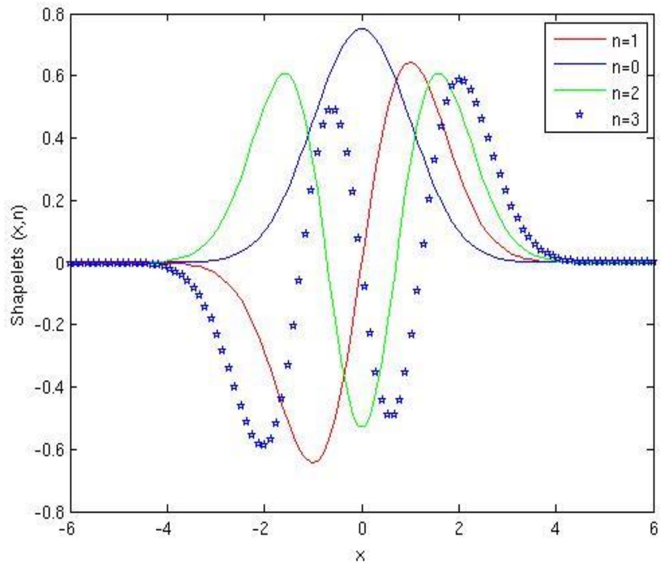


Fig.4

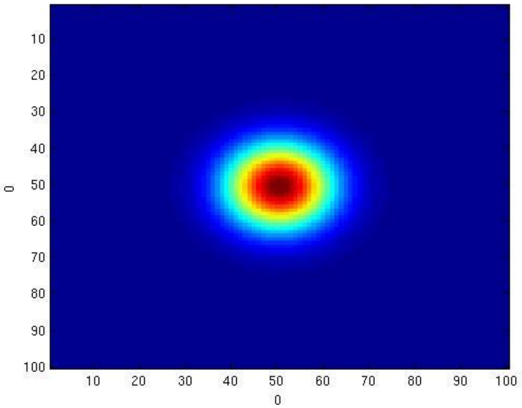
1)Cartesian system: Shapelets for 2D

Extending the 1D Shapelet formalism to 2D Shapelet formalism.

Given an (x1,x2) Cartesian grid we can define in every point (x1,x2) the 2D Shapelts:

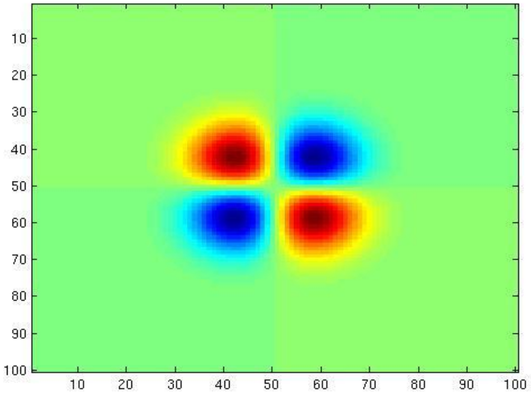
$$\phi_{\mathbf{n}}(\mathbf{x}) \equiv \phi_{n_1}(x_1)\phi_{n_2}(x_2),$$

Observation : This basis is again orthonormal.



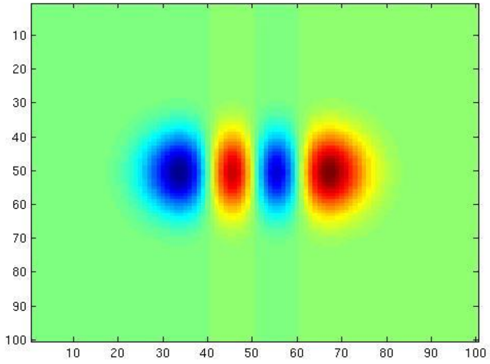
n1=0;n2=0

Fig.5



n1= 1; n2= 1

Fig.6



n1=3; n2=0

Fig.7

2)Polar system:
Hermite polar polynomials;1D Polar shapelets

1)Hermite polar polynomials

Recurrence relation :

$hpr(k,l,x) = (x / (l - k)) * l * hpr(k,l - 1,x) - (x / (l - k)) * k * hpr(k - 1,l,x)$

2) The Laguerre polynomials can be defined in two ways:

2.1) $Lag(z^2,k,l) = hpr(k,l,z) * (1 / (((-1)^k) * k! * (z^{(}l - k))))$

2.2) $Lag1(k,l\text{-}k,x) = (2 + ((l\text{-}k) - 1 - x) / k).* Lag1(k - 1,(l\text{-}k),x) - (1 + ((l\text{-}k) - 1) / k)*Lag1(p - 2,(l\text{-}k),x);$

Observation: Results are consistent:
for k=1;l=2;
> z1=laguerre(x,1,2);
>> >> z2=laguerre1(1,1,x);
>> z1./z2
ans = Columns 1 through 7
1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000

3)The polar 1d Basis function:

$$X(k,l,x,phi) = 1/\sqrt{(\pi * k!l!)} * hpr(k,l,x) * \exp(-x^2/2) * \exp(i(l-k) * \phi)$$

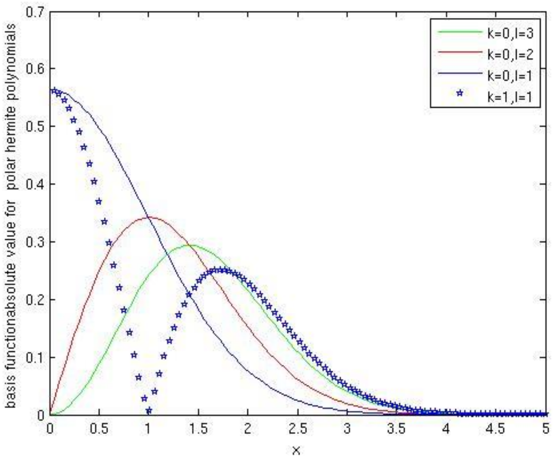


Fig.10

2)Polar system:
Polar 2D Shapelets

- The basis function for 2D polar shapelets:
- Usual coordinates transformations in a Cartesian grid:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \mathrm{atan2}(y, x)$$

$$\chi_{n,m}(r,\theta;\beta) = \frac{(-1)^{\frac{n-|m|}{2}}}{\beta^{|m|+1}} \left[\frac{\left(\frac{n-|m|}{2}\right)!}{\pi \left(\frac{n+|m|}{2}\right)!} \right]^{\frac{1}{2}} \times r^{|m|} L^{|m|}_{\frac{n-|m|}{2}} \left(\frac{r^2}{\beta^2} \right) e^{\frac{-r^2}{2\beta^2}} e^{-im\theta} .$$

m=0;
n=0;

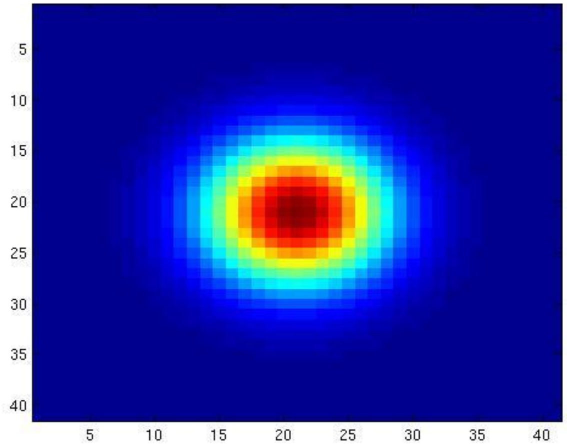
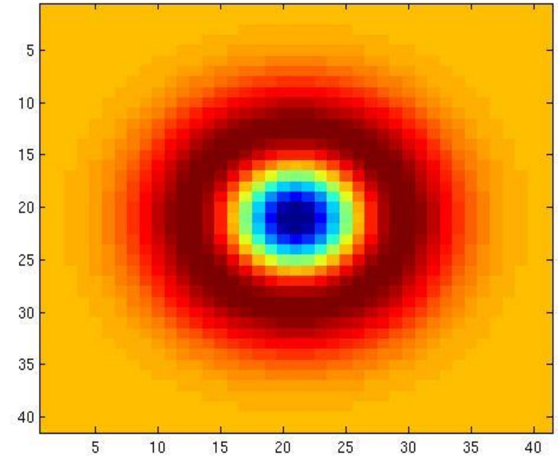


Fig.11

Relations between the coefficients:
m=l-k
n=l+k



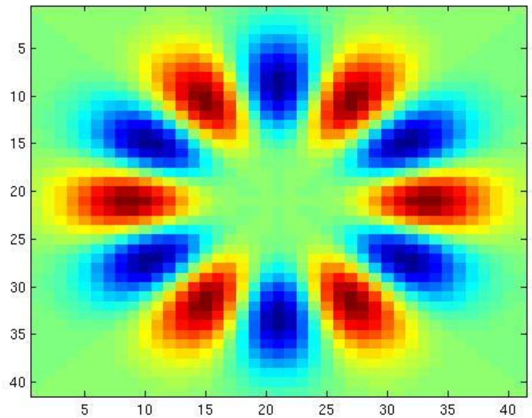
m=1;
n=1;

Fig.12

2)Polar system:
Polar 2D Shapelets

Real

k=0
l=6

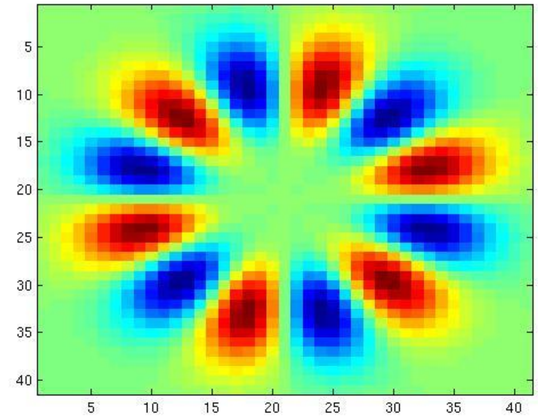


m=6
n=6

Fig.13

Imaginary

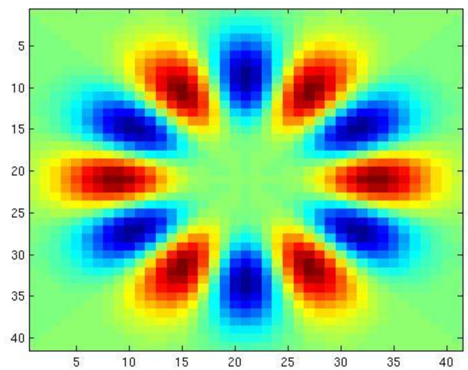
k=0
l=6



m=6
n=6

Fig.14

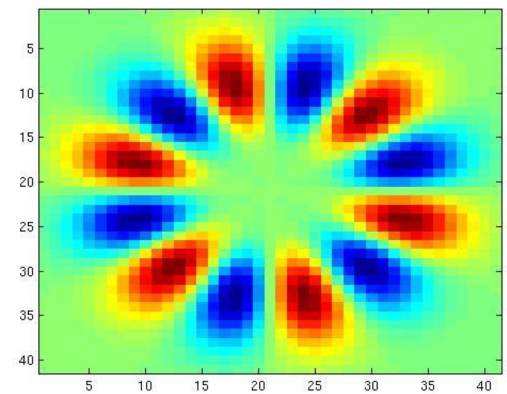
k=6
l=0



m=-6
n=6

Fig.15

k=6
l=0



m=-6
n=6

Fig.16

3)Creating images using Cartesian and Polar Shapelets

- With the basis functions we can create images:

$$f(r,\theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{n,m} \chi_{n,m}(r,\theta;\beta) .$$

- 1)Cartesian Shapelets for creating an image:
- a)Random scalars

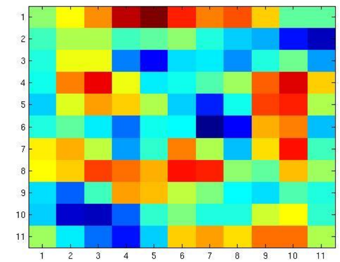


Fig.17

b) A(1,1)=1 ;

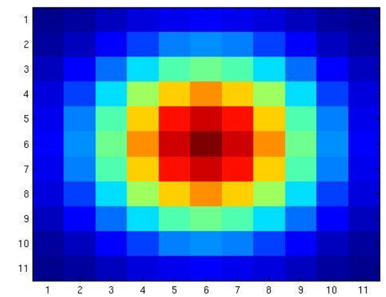
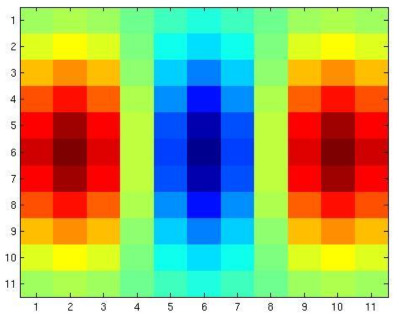
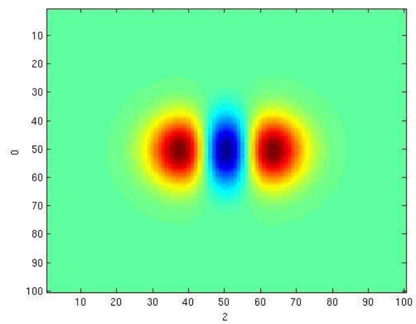


Fig.18



A(3,1)=1

Fig.19



m=2; n=0;

Fig.20

A(2,2)=1

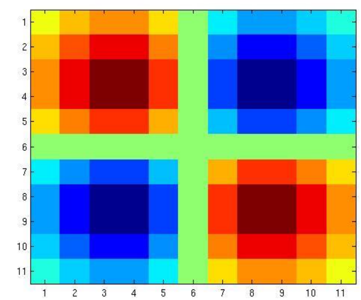


Fig.21

m=1; n=1;

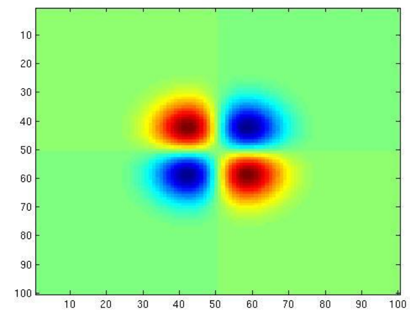


Fig.22

3)Creating images using Cartesian and Polar Shapelets

A(1,1)=1;
A(2,1)=3;

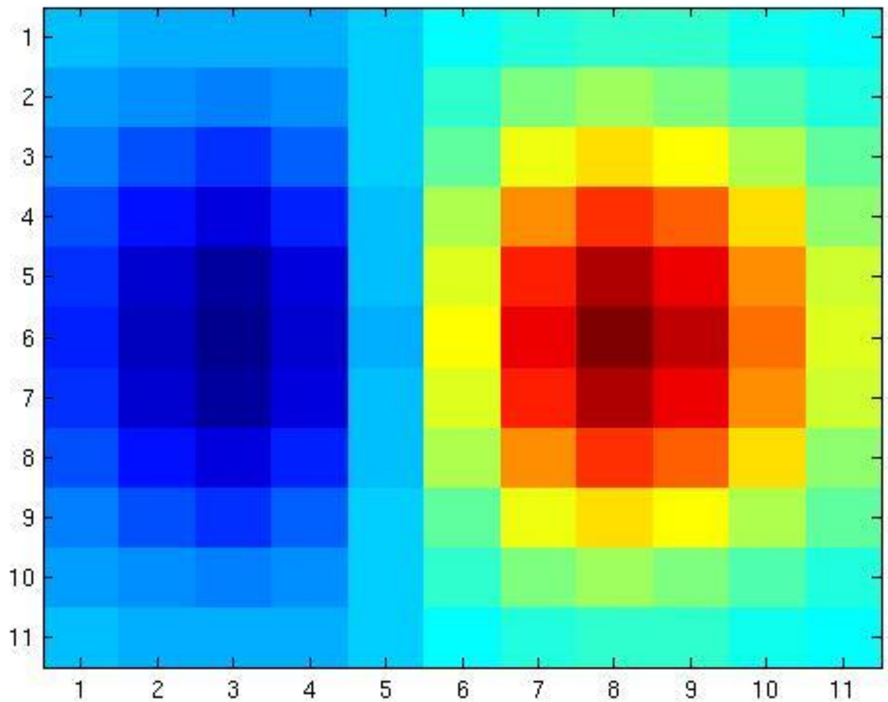


Fig.23

3)Creating images using Cartesian and Polar Shapelets

- Polar shapelets

random

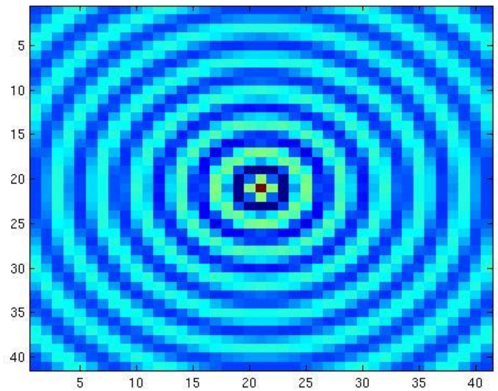


Fig.24

A(1,1)=1

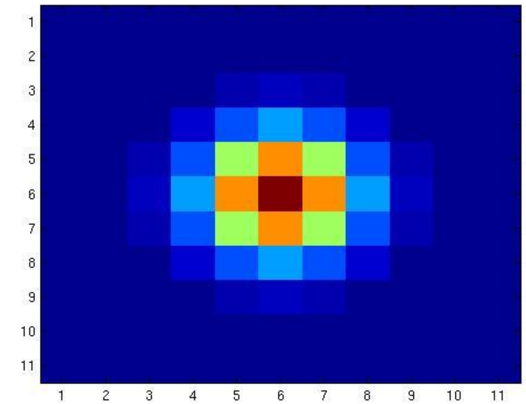


Fig.25

A(3,3)=1

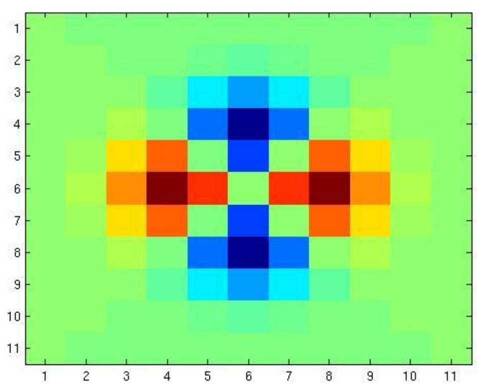


Fig.26

m=2;
n=2;

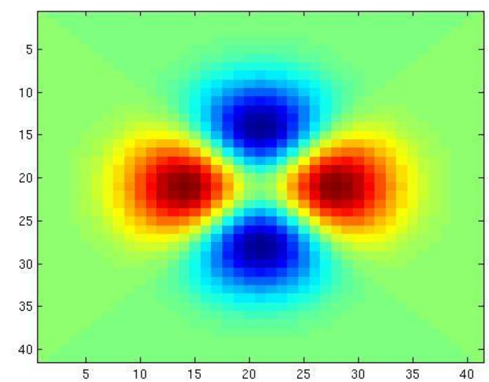


Fig.27

Next steps :

1)Optimization of the polynomials codes.

2)Create a function to compute the forward shapelet transform by approximating the integral (i.e. from an image to shapelet coefficients);

$$f(r,\theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{n,m} \chi_{n,m}(r,\theta;\beta) \, . \qquad \rightarrow \qquad f_{n,m} = \iint_{\mathbb{R}} f(r,\theta) \, \chi_{n,m}(r,\theta;\beta) \, r \, \mathrm{d}r \mathrm{d}\theta \, .$$

3)Recover shapelet coefficients from a noisy simulation

- Simulate an image with known shapelet coefficients
- Add noise
- Recover the shapelet coefficients by forward transform
- Recover the shapelet coefficients by l1 minimisation (i.e. compressive sensing recovery);

Epilogue

Thank you for your attention!