Differential forms, Integration and the Degree Theorem

MSc student Vlad Margarint, Mathematics Department, ETH Zürich

This material represents a continuation of the ideas that were developed during the semester on the Seminar "Topology from the differentiable viewpoint" organized at ETH Zürich by Felix Hensel. The text contains a brief introduction to *differential forms* in Euclidean space and in the context of manifolds, along with their basic properties. The second part of the material contains two main results regarding integration on manifolds: Stokes Theorem and the Degree Theorem.

We will start by introducing the differential forms and to study their basic properties. The first concept is the definition of an alternating k-form. Keep in mind that a familiar example of alternating k-form is the determinant of a matrix. In general:

Definition 1. *Let S_k be the permutation group on k elements, i.e. the group of all bijective maps $\sigma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$.Let $\epsilon(\sigma)$ be the signature of the permutation σ. An alternating k-form on a vector space V is a multilinear map $\omega : \underbrace{V \times V \times \ldots \times V}_{k\ times} \to \mathbb{R}$ satisfying:*

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \epsilon(\sigma)\omega(v_1, \ldots, v_k), \qquad \forall\ v_1, \ldots, v_k \in V, \forall\ \sigma \in S_k.$$

The vector space of all alternating k-forms on V will be denoted by:

$$\Lambda^k V^* := \{\omega : V^k \to \mathbb{R} |\ \omega \textit{ is an alternating } k\textit{-form}\}.$$

Moreover, we can construct new alternating forms using previous ones:

Definition 2. *Let $k,\ l \in \mathbb{N}$. We define the set of all $(k,l)-$ shuffles $S_{k,l}$, i.e. $S_{k,l} \subset S_{k+l}$ is the set of all permutations that leave the order of the first k elements and of the last l elements unchanged:*

$$S_{k,l} := \{\sigma \in S_{k+l} | \sigma(1) < \sigma(2) < \sigma(3) \ldots < \sigma(k),\ \ \sigma(k{+}1) < \sigma(k{+}2) < \ldots < \sigma(k{+}l)\}.$$

Definition 3. *The exterior product of two alternating forms $\omega \in \Lambda^k V^*$ and $\tau \in \Lambda^k V^*$ is the $k+l$ form $\omega \wedge \tau \in \Lambda^{(k+l)} V^*$ defined by:*

$$(\omega \wedge \tau)(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in S_{k,l}} \epsilon(\sigma) \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}), \quad \forall\ v_1, \ldots, v_{k+l} \in V.$$

Another procedure of constructing new alternating forms is using a linear map between two vector spaces :

Definition 4. *Let $\phi : V \to W$ be a linear map. The pull-back of an alternating k-form $\omega \in \Lambda^k W^*$ is the alternating k-form $\phi^* w$ on V defined by:*

$$(\phi^* \omega)(v_1, \ldots, v_k) := \omega(\phi(v_1), \ldots, \phi(v_k)), \quad \forall\ v_1, \ldots, v_k \in V$$

Remark 1. *We will use in the proof of the main results of the material the following identity that represents in fact the generalization of the change of variables rule in one-dimensional calculus. In the differential forms language the identity can be written as follows :*

If $\phi : V \to V$ is an automorphism and $\omega \in \Lambda^m V^*$, where m=dim(V), then we have that $\phi^* \omega = det(\phi) \omega$.

In the study of Differential Topology on manifolds we will define a special family of alternating k-forms for which the domain of the definition is not a general vector space but the tangent space at a given point on a manifold.

Definition 5. *A differential k-form on M is a collection of alternating k-forms (observe the index $p \in M$, so for each point on the manifold you can define one) $\omega_p \in T_pM \times T_pM \ldots \times T_pM \to \mathbb{R}$ such that $\forall$ k-tuple $X_1, \ldots, X_k$ of vector fields the function $p \mapsto \omega_p(X_1(p), \ldots, X_k(p))$ is smooth.*

A differential k-form ω is said to have *compact support* if the set:

$$supp(\omega) := \overline{\{p \in M | \omega_p \neq 0\}}$$

(called support of ω) is compact. We consider $\Omega^k(M)$ to be the set of all k-forms on M and $\Omega_c^k(M)$ the set of all k-forms with compact support. The concept of exterior product and pull-back can be defined in the context of smooth manifolds exactly like they were defined before. Due to the fact that the definition is dependent on the point the pointwise exterior product defines a bilinear map: $\Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$

$$(\omega, z) \mapsto \omega \wedge z$$

such that for $p \in M$ we have $(\omega \wedge z)_p := \omega_p \wedge z_p$.

In addition, given a smooth map between manifolds $f : M \to N$ and a k-form ω on N we can define the pullback $f^*\omega$ under f as the differential k-form on M via:

$$(f^*\omega)_p(v_1, \ldots, v_k) := \omega_{f(p)}(df(p)v_1, \ldots, df(p)v_k), \quad \forall p \in M, v_1, \ldots, v_k \in T_pM.$$

Another important tool in the study of differential forms is called the exterior differential. Applying the exterior differential operator on a k-form we obtain a k+1 form. The definition of such an operation is formulated in Euclidean space but we will define it directly for differential forms on manifolds (for the general definition consult [3]).

Definition 6. *Let M be a m-dimensional manifold. We call $\{U_i, \phi_i\}_{i \in I}$ an atlas of M if it has the following properties:*

1)$\bigcup\limits_{i \in I} U_i = M$.

2)Each map $\phi_i : U_i \to \phi_i(U_i)$ is a homeomorphism onto an open set of $\mathbb{R}^m$ such that the transition maps:

$$\phi_{ji} : \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

are smooth.

Definition 7. *Let $\omega \in \Omega^k(M)$. The exterior differential of ω is the $(k+1)$ form $d\omega \in \Omega^{k+1}(M)$ defined by:*

$$d\omega(x; \xi_1, \ldots, \xi_{k+1}) := \sum_{j=1}^{k+1} (-1)^{j-1} \frac{d}{dt}\bigg|_{t=0} \omega(x + t\xi_j; \xi_1, \ldots, \widehat{\xi_j}, \ldots, \xi_{k+1})$$

for $x \in U$ and $\xi_1, \xi_2, \ldots, \xi_{k+1} \in \mathbb{R}^m$. The hat indicates that the j^{th} term is deleted.

Moreover, we call the differential form ω:

*1)***closed** *if* $d\omega = 0$.

*2)***exact** *if* $\exists\, \eta \in \Omega^{k-1}(M)$ *such that* $d\eta = \omega$.

Definition 8. *Let* M *be an* m*-dimensional manifold with an atlas* $(U_i, \phi_i)_{i\in I}$ *and* $\omega \in \Omega^k(M)$ *a differential* k*-form on* M*. Using the coordinate map we set:*

$$\omega_i \in \Omega^k(\phi_i(U_i) \qquad s.t. \qquad \omega|_{U_i} = \phi_i^*(\omega_i) \quad \forall i \in I.$$

The exterior differential of ω *is the unique differential (k+1) form* $d\omega \in \Omega^{k+1}(M)$ *that satisfies:*

$$d\omega|_{U_i} = \phi_i^* d\omega_i, \quad \forall i \in I.$$

Remark 2. *The most important properties that we are going to use in the proofs of the theorems are:*

1)Pull-back and the exterior product commute i.e.:

$$f^*(\omega \wedge z) = f^*\omega \wedge f^*z, \quad \forall f : M \to N \;\; smooth, \quad \forall z, \omega \in \Omega^k(N).$$

2)Pull-back and the exterior derivative commute i.e.:

$$f^* d\phi = d(f^*\phi), \quad \forall f : M \to N \;\; smooth, \quad \forall \phi \in \Omega^k(M).$$

In the study of integration on manifolds we need several tools. Firstly, we will need a precise definition about what integration on manifolds represents and after that we will try to deduce important results using the properties discussed in the beginning of the material. Before we start, we have to be careful about the definition of the atlas in this material. In this material U_i will be subsets of the manifold that are homeomorphic with open sets of the upper half space $\mathbb{H}^m := \{(x^1, x^2, x^3, \dots, x^m) \in \mathbb{R}^m | x^m \geq 0\}$.

Let M be an oriented m- manifold M (with or without boundary, compact or non-compact) and $\{U_i, \phi_i\}_{i\in I}$ a (w.l.o.g.) positively oriented atlas of M (i.e. the transition maps $\phi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ are smooth and $\det(d\phi_{ij}(x)) > 0 \;\forall i, j \in I$). We can choose a partition of unity $\rho_i : M \to [0, 1]$, $i \in I$ subordinate to the open cover given by the atlas. Note that the partition of unity exists also if the manifold is not oriented. In this setting we can define rigorously the integration on a manifold:

Definition 9. *Let* ω *be a differential* m*-form on the manifold* M *with compact support (i.e.* $\omega \in \Omega_c^m(M)$*) and* $\omega_i \in \Omega^m(\phi_i(U_i))$, $g_i : \phi_i(U_i) \to \mathbb{R}$ *smooth, such that* $\omega|_{U_i} = \phi_i^*\omega_i$*, where:*

$$\omega_i =: g_i(x) dx^1 \wedge dx^2 \wedge dx^3 \dots \wedge dx^m, \forall i \in I.$$

The integral of ω over M is the real number:

$$\int_M \omega = \sum_{i\in I} \int_{\phi_i(U_i)} \rho_i(\phi_i^{-1}(x))g_i(x)dx^1 \dots dx^m.$$

The first remark that one should make is that the sum defined on the right hand side is finite due to the fact that the partition of unity that we have chosen it is subordinate to the cover given by the atlas. Since ω is compactly supported, there are only finitely many $i \in I$ such that g_i is non-vanishing.

It the following we will prove that the integration procedure is well defined, i.e. the result of the integration does not depend on the way we choose the atlas and the subordinate partition of unity.

Lemma 1. *Let $(U_i, \phi_i)_{i\in I}$ and $(V_j, \psi_j)_{j\in J}$ two independent oriented atlases of M, and $\{\rho_i\}_{i\in I}$ and $\{\theta_j\}_{j\in J}$ the partitions of unity subordinate to the two coverings of M. Then the integral of ω over M is independent of the two oriented atlases.*

Proof. For the first atlas we keep the notations like in the brief description before the lemma.

In the second atlas following the definitions above we define: $\omega_j \in \Omega^m(\psi_j(V_j))$ by $\omega_j =: h_j(x)dx^1 \wedge dx^2 \wedge dx^3 \dots \wedge dx^m, \forall j \in J$, where $\omega|_{V_j} = \psi_j^* \omega_j$ and $h_j : \psi_j(V_j) \to \mathbb{R}$.

From remark 1 it follows that:

$$g_i(x) = h_j(\psi_j \circ \phi_i^{-1}(x))det(d(\psi_j \circ \phi_i^{-1})(x)), \ \ \forall z \in \phi_i(U_i \cap U_j).$$

Due to the fact that both atlases are positively oriented then the value $det(d(\psi_j \circ \phi_i^{-1})(x)) > 0$. Using the fact that $\{\rho_i\}_{i\in I}$ and $\{\psi_j\}_{j\in J}$ are two partitions of unity and change of the variable formula in integration we obtain that:

$$\begin{aligned}
\textstyle\int_M \omega &= \sum_{i\in I} \textstyle\int_{\phi_i(U_i)} (\rho_i \circ \phi_i^{-1}) g_i dx^1 dx^2 \dots dx^m \\
&= \sum_{i\in I} \sum_{j\in J} \textstyle\int_{\phi_i(U_i\cap V_j)} (\rho_i \circ \phi_i^{-1})(\theta_j \circ \phi_i^{-1}) g_i dx^1 dx^2 \dots dx^m \\
&= \sum_{i\in I} \sum_{j\in J} \textstyle\int_{\psi_j(U_i\cap V_j)} (\rho_i \circ \psi_j^{-1})(\theta_j \circ \phi_j^{-1}) h_j dy^1 dy^2 \dots dy^m \\
&= \sum_{j\in J} \textstyle\int_{\psi_j(V_j)} (\theta_j \circ \psi_j^{-1}) h_i dy^1 dy^2 \dots dy^m.
\end{aligned}$$

□

Remark 3. *If the two atlases from the previous lemma were reverse oriented then the proof will be the same except that the result will have the opposite* *sign. More generally, if $f : M \to N$ is a diffeomorphism between oriented m-manifolds then $\int_M (f^*\omega) = +\int_N \omega,\ \forall \omega \in \Omega_c^m(N)$, if f is orientation preserving and $\int_M (f^*\omega) = -\int_N \omega,\ \forall \omega \in \Omega_c^m(N)$, if f is orientation reversing.*

One important result in the integration over manifolds is the Theorem of Stokes that will be presented in the following. The proof is composed of three steps and it uses Fubini's theorem for multiple integration.

Theorem 1. *Let M be an oriented m-manifold with boundary and let $\omega \in \Omega_c^{m-1}(M)$. We have the following equality:*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. Step 1)First, we will prove the result for M=$\mathbb{H}^m$. The boundary of this domain is exactly the set of points $(x^1, x^2, \ldots, x^m) \in \mathbb{R}^m$ that satisfy $x^m = 0$. This set is clearly diffeomorphic to $\mathbb{R}^{m-1}$. Consider $g_i : \mathbb{H}^m \to \mathbb{R}$ smooth functions with compact support in the upper half space. We take the differential $(m-1)$ form:

$$\omega = \sum_{i=1}^{m} g_i(x) dx^1 \wedge dx^2 \wedge dx^3 \ldots, \wedge \widehat{dx^i} \ldots \wedge dx^m,$$

where the "hat" means that the corresponding term is deleted in the i^{th} summand. Using the definition of the exterior derivative we obtain that :

$$d\omega = \sum_{i=1}^{m} \frac{\partial g_i}{\partial x^i} dx^i \wedge dx^1 \ldots \wedge dx^m.$$

Using the commutativity relation for the wedge product we obtain that:

$$d\omega = \sum_{i=1}^{m} (-1)^{i-1} \frac{\partial g_i}{\partial x_i} dx^1 \wedge dx^2 \ldots \wedge dx^m.$$

Due to the fact that g_i has compact support $\forall i \in I$ choose $R > 0$ large enough such that the support of each g_i is contained in the "box" $[-R, R]^{m-1} \times [0, R]$. From Fubini's theorem and the remark that the value of the differential form on the boundary is given by:

$$\omega|_{\partial \mathbb{H}^m} = g_m(x^1, x^2, \ldots, x^{m-1}, 0) dx^1 dx^2 \ldots dx^{m-1}.$$

It follows:

$$\int_{\mathbb{H}^m} d\omega = \sum_{i=1}^{m}(-1)^{i-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R \frac{\partial g_i}{\partial x_i}(x^1,x^2,\ldots,x^m)dx^1\ldots dx^m$$

$$= (-1)^{m-1}\int_{-R}^R\cdots\int_{-R}^R\int_0^R \frac{\partial g_m}{\partial x_m}(x^1,x^2,\ldots,x^{m-1},x^m)dx^m dx^1\ldots dx^{m-1}$$
$$= (-1)^m\int_{-R}^R\cdots\int_{-R}^R g_m(x^1,\ldots,x^{m-1},0)dx^1\ldots dx^{m-1}$$
$$= \int_{\partial\mathbb{H}^m} w.$$

The last but one equality is a consequence of the Fundamental Theorem of Calculus. To make it more precise, only the last term remains in the equality when Fubini's theorem is used because all the integrals up to order m are vanishing. The reason for that is the Fundamental Theorem of Calculus. Moreover, because the outward pointing unit normal at the boundary of $\mathbb{H}^m$ is $n = (0,0,0,\ldots,-1)$ it follows that the orientation of $\mathbb{H}^m$ is $(-1)^m$ times the orientation of the standard $\mathbb{R}^{m-1}$.

Step 2: *We will prove the same result with the condition that the* $(m-1)$ *differential form has its compact support contained in a coordinate chart.*

Let's consider $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{H}^m$ a coordinate chart for the manifold M and $\omega \in \Omega_c^{m-1}(M)$ with $supp(\omega) \subset U_i$.

We can define $\omega_i \in \Omega^{m-1}(\phi_i(U_i))$ by $\omega|_{U_i} = \phi_i^*\omega_i$. Using this procedure we can extend ω_i to all of $\mathbb{H}^m$ by setting it to be zero on the complement of $\phi_i(U_i)$ in $\mathbb{H}^m$. On the boundary we have that $\phi_i(U_i \cap \partial M) = \phi_i(U_i) \cap \partial\mathbb{H}^m$. Using *Step 1* we obtain that:

$$
\begin{aligned}
\int_M d\omega &= \int_{U_i} d(\phi_i^* \omega_i) \\
&= \int_{U_i} \phi_i^* d\omega_i \\
&= \int_{\phi_i(U_i)} d\omega_i \\
&= \int_{\phi_i(U_i)\cap\partial\mathbb{H}^m} \omega_i \\
&= \int_{U_i\cap\partial M} \phi_i^* \omega_i \\
&= \int_{\partial M} \omega.
\end{aligned}
$$

The sequence of equalities it is motivated, in this order, by the the properties that we studied for the pull-back of the differential forms, change of variable formula, the formula that we derived in *Step 1*, and using again the change of the variable formula.

Step 3): We are going to prove the theorem in its general form. For this, choose an atlas $(U_i, \phi_i)_{i\in I}$ and a partition of unity $\rho_i : M \to [0, 1]$ subordinate to this atlas. By *Step 2* the following sequence of equalities holds:

$$
\begin{aligned}
\int_M d\omega &= \sum_i \int_M d(\rho_i \omega) \\
&= \sum_i \int_{\partial M} \rho_i \omega \\
&= \int_{\partial M} \omega.
\end{aligned}
$$

The last step proves the result in its generality. □

The theorem of Stokes has many applications to problems that arise from Physics. Under certain conditions, it relates the value of a surface integral of a function with the one computed on the boundary for another function. The result has a lot of applications in many problems both abstract and concrete like: measuring the heat flux of a given heat source on a given surface (for example, that heat source can be a star) or measuring the water pressure on the boundary of a cylindrical tube.

We apply the Theorem of Stokes in the following examples:

Example 1. *1)Stokes theorem in the 2 dimensional Euclidean space has the following form: Consider* $f, g : \mathbb{R}^2 \to \mathbb{R}$ *two smooth functions and the following differential form:* $\omega = f dx + g dy \in \Omega^1(\mathbb{R}^2)$. *Consider a domain* $U \in \mathbb{R}^2$ *with boundary* ∂U *given by a positively oriented, piece-wise smooth, simple closed curve. Then using the properties that we presented in the first part of the material we deduce that* $d\omega = (\partial g/\partial x - \partial f \partial y) dx \wedge dy$. *Finally, using Stokes Theorem we deduce the following equality:*

$$\int_U (\partial g/\partial x - \partial f/\partial y) dx \wedge dy = \int_{\partial U} (f dx + g dy).$$

2)If M *is an oriented m-manifold without boundary and* $\epsilon \in \Omega_c^{m-1}(M)$ *is a compactly supported* $(m-1)$ *form it follows from Stokes theorem that:*

$$\int_M d\epsilon = 0.$$

Example 1.2) is the starting point in the study of the next result: *The Degree Theorem.* First, we will study a result about integration and exactness. If M is a connected manifold there is a result conversely to the one in Example 1.2) given by the following theorem:

Theorem 2. *Given* M *a connected oriented m-dimensional manifold without boundary and* $\omega \in \Omega_c^m(M)$ *then the integral over* M *of* ω *vanishes if and only if there is an* $(m-1)$ *form* τ *on* M *with compact support such that* $d\tau = \omega$.

Note: The proof will use a corollary of Cartan's Theorem that is not going to be proven in this material. For a complete proof of the result consult [3]. Before introducing the theorem of Cartan we will introduce the necessary concepts:

Let M and N be two smooth manifolds, $I \subset \mathbb{R}$ be an interval and

$$I \times M \to N : (t, p) \mapsto \phi_t(p)$$

be a smooth map. For $t \in I$ we define the operator

$$h_t : \Omega^k(N) \to \Omega^{k-1}(M)$$

by

$$(h_t \omega)_p(v_1, v_2, \ldots, v_{k-1}) = \omega_{\phi_t}(p)(\partial_t \phi_t(p), d\phi_t(p) v_1, \ldots, d\phi_t(p) v_{k-1}), \quad \forall p \in M, v_i \in T_p M.$$

Cartan's Theorem states the following:

Theorem 3. *For every differential k form $\omega \in \Omega^k(N)$ with the notations as before we have that:*

$$\frac{d}{dt}\phi_t^*\omega = dh_t\omega + h_t d\omega.$$

In the proof of Theorem 2 we will use the following corollary of Cartan's theorem :

Corollary 1. *Given M and N two oriented manifolds without boundary and $\phi_t : M \to N$ with $t \in [0, 1]$ be a proper smooth homotopy (i.e. a map that reverts compact subsets of the codomain into compact subsets of the domain and in addition is a smooth homotopy) so that: given $K \subset N$ compact $\Rightarrow \bigcup_t \phi_t^{-1}(K) \subset M$ is compact. For every $\omega \in \Omega_c^k(N)$ closed k-form there exists a $(k-1)$ form $\tau \in \Omega_c^k(N)$ with the following property:*

$$d\tau = \phi_1^*\omega - \phi_0^*\omega.$$

The proof of the corollary is just a simple application of Cartan's Theorem:

Proof. Taking the operator $h_t : \Omega^k(M) \to \Omega^{k-1}(M)$ defined before we obtain:

$$\begin{aligned}\phi_1^*\omega - \phi_0^*\omega &= \int_0^1 \frac{d}{dt}(\phi_t^*\omega)dt \\ &= \int_0^1 d(h_t\omega)dt \\ &= d\tau.\end{aligned}$$

In the last equation we took $\tau := \int_0^1 h_t\omega dt$. □

We are now ready to prove Theorem 2 :

Proof. $\Leftarrow$ This part of the theorem is a direct consequence of Stokes's theorem.

$\Rightarrow$ This proof is going to be done in two steps:

Step 1) Let $f : \mathbb{R}^m \to \mathbb{R}$ whose support is contained in the cube $(a, b)^m$. Then there are smooth functions $v_i : \mathbb{R}^m \to \mathbb{R}$, $\forall i = 1, 2, \ldots m$ such that $f = \sum_{i=1}^{m} \frac{\partial v_i}{\partial x_i}$. Given the properties of the exterior derivative we have that the differential m form can be expressed as:

$$f dx^1 \wedge dx^2 \wedge dx^3 \ldots \wedge dx^m = d(\sum_{i=1}^{m}(-1)^{i-1}u_i dx^1 \wedge dx^2 \ldots \widehat{dx^i} \wedge dx^m).$$

This can be proven using the existance of the so called bump function. Their existance guarantees that we can use a smooth function with $\rho : \mathbb{R} \to [0,1]$ which satisfies:

$$\rho(t) = \begin{cases} 0, \text{ for } t \leq a + \epsilon \\ 1, \text{ for } t \geq b - \epsilon. \end{cases}$$

Define $f_i : \mathbb{R}^m \to \mathbb{R}$ by $f_0 = 0$ and $f_m = f$. For all $i = 1, 2, \ldots, m-1$ we will conveniently define the following smooth functions with compact support in $(a, b)^m$(this will follow directly from the definition of ρ and f_i):

$$f_i = \int_a^b \cdots \int_a^b f(x^1, x^2, \ldots, x^i, \omega^{i+1}, \omega^{i+2}, \ldots, \omega^m)\dot{\rho}(x^{i+1}) \ldots \dot{\rho}(x^m) d\omega^{i+1} \ldots d\omega^m.$$

Considering the previous definition we will our function u_i such that their partial derivatives sum up to f: $u_i := \int_a^{x^i} (f_i - f_{i-1})(x^1, \ldots, x^{i-1}, \omega, x^i, \ldots, x^m) d\omega$.

From the definition of f_i and f_{i-1} we have that u_i is compactly supported. In addition, from the Fundamental Theorem of Calculus we obtain the desired equality: $\frac{\partial u_i}{\partial x^i} = f_i - f_{i-1}$.

Step 2)In the second step we will prove the $\Rightarrow$ implication. Fix a point $p_0 \in M$ and take an open neighborhood $U_0 \subset M$ containing p_0. We can consider an orientation preserving coordinate chart such that $\phi_0(U_0) = (0,1)^m$ the m-dimensional open cube in $\mathbb{R}^m$. Now, the condition that M is connected and has no boundary becomes essential because for every $p \in M$ we can construct a diffeomorphism $\psi_p : M \to M$ such that $\psi_p(p_0) = p$ and in addition ψ_p is isotopic with the identity. Varying $p \in M$ we obtain an open cover of the manifold(namely the open sets of the cover are $U_p := \psi_p(U_0)$. Next, we choose a partition of unity $\{\rho_p\}_{p \in M}$ subordinate to this special cover. In the hypothesis we assumed that the support of ω is compact so there are only finitely many points $p \in M$ such that $\rho_p \neq 0$ on $supp(\omega)$ which will be denoted by $p_1, p_2, \ldots, p_n$. We will abbreviate for simplicity in the following way: $U_i := U_{p_i}$; $\rho_i := \rho_{p_i}$; $\psi_i := \psi_{p_i}$. It is clear that, $supp(\rho_i) \subset U_i$ and $\sum_{i=1}^{n} \rho_i|_{supp(\omega)} = 1$. Using the fact that the partition of unity is suboridinated to this particular cover we have that $supp(\rho_i\omega) \subset U_i$. Using the definition of the pull-back it follows that $supp(\psi_i^*(\rho_i\omega)) \subset U_0$. Now we can use the fact that ψ_i is homotopic to the identity (so it respects the conditions of the Corollary of Cartan's Theorem) and $\phi_i\omega$ has compact support to obtain that exists a compactly supported $(m-1)$ form $\tau_i \in \Omega_c^{m-1}(M)$ such that $d\tau_i = \psi_i^*(\rho_i\omega) - \psi_0^*(\rho_i\omega)$. Using the fact that the map is homotopic to the identity (so ψ_0^* is equal to the identity) we obtain that:

$$\int_M \sum_{i=1}^n \psi_i^*(\rho_i\omega) = \int_M \sum_{i=1}^n (d\tau_i + \rho_i\omega).$$

Using the fact that τ_i is compactly supported and Stokes Theorem together with $\partial M = 0$ we obtain that:

$$\int_M \sum_{i=1}^n d\tau_i + \rho_i\omega = \int_M \sum_{i=1}^n \rho_i\omega = \int_M \omega = 0.$$

Now, we will use our coordinate chart to evaluate our integral in $\mathbb{R}^m$. Using the fact that $\psi_i^*(\rho_i\omega)$ (and also $\sum_{i=1}^n \psi_i^*(\rho_i\omega)$)has its support in $\psi_i^{-1}(U_i) = U_0$ and that $\phi_0(U_0) = (0,1)^m$ we can take the pushforward of the sum using the chart ϕ_0 denoted by $(\phi_0)_* \sum_{i=1}^n \psi_i^*(\rho_i\omega)$ (i.e. the sum of the corresponding elements in M via ϕ_0) and we obtain that:

$$\int_{\mathbb{R}^n} (\phi_0)_* \sum_{i=1}^n \psi_i^*(\rho_i\omega) = \int_M \sum_{i=1}^n \psi_i^*(\rho_i\omega) = 0.$$

The previous equality is obtained using the fact that our functions can be smoothly extended by setting it equal to 0 on $\mathbb{R}^m \setminus (0,1)^m$.

Using *Step 1* there is an $(m-1)$ form $\tau_0 \in \Omega_c^{m-1}(\mathbb{R}^m)$ with support in $(0,1)^m$ such that:

$$d\tau_0 = (\phi_0)_* \sum_{i=1}^n \psi_i^*(\rho_i\omega).$$

It follows that $\phi^*\tau_0 \in \Omega_c^{m-1}(U_0)$ has compact support in U_0 and therefore extends to all of M by setting it to be equal to zero on $M \setminus U_0$. This extension satisfies $d\phi_0^*\tau_0 = \sum_{i=1}^n \psi_i^*(\rho_i\omega)$ and because of that we have the following sequence of equalities:

$$\omega = \sum_{i=1}^n \psi_i^*(\rho_i\omega) - \sum_{i=1}^n (\psi_i^*(\rho_i\omega) - \rho_i\omega) = d\phi_0^*\tau_0 - \sum_{i=1}^n d\tau_i = d(\phi_0^*\tau_0 - \sum_{i=1}^n \tau_i).$$

The previous equality shows the existence of the desired differential form, namely $\tau := \phi_0^*\tau_0 - \sum_{i=1}^n \tau_i$.

□

The last result of this material is the *Degree Theorem* that relates the value of the integral of a differential form and the integral of its pull-back with an arbitrary smooth map between two compact, oriented manifolds without boundary of the same dimension. The theorem holds if given an arbitrary smooth map $f : M \to N$ we require N to be connected. Firstly, we have to define the degree of a map using the following result:

Lemma 2. *Let $f : M \to N$ be a smooth map between two m-dimensional compact manifolds without boundary. If $q \in N$ is a regular value of f then $f^{-1}(q)$ is finite in M.*

Proof. The set $f^{-1}(q)$ is a closed set in M due to the fact that is the preimage of the closed set $\{q\} \in N$. Due to the fact that any closed subset of a compact topological space is compact it follows that $f^{-1}(q)$ is compact. Using the fact that q is a regular value it follows that the differential of f is surjective between two tangent spaces of the same dimension we obtain that f is a local diffeomorphism. Using the inverse function theorem we obtain that $f^{-1}(y)$ is discrete. Due to the fact that a discrete and compact set is finite we obtain the conclusion.

□

Now, we can denote the elements of $f^{-1}(q)$ by $\{p_1, p_2, \ldots, p_n\}$. We consider $\epsilon_i = \pm 1$ according to the orientation preserving or reversing of the differential $df(p_i) : T_{p_i}M \to T_qN$ (i.e. $\epsilon_i = sign(det(df(p_i)))$). Now, we will define the degree of f to be:

$$deg(f) := \sum_{i=1}^{n} \epsilon_i.$$

The following result is the *Degree Theorem* :

Theorem 4. *Let M and N be compact oriented smooth m-manifolds without boundary and in addition N is connected. It follows that for every smooth map $f : M \to N$ and every $\omega \in \Omega^m(N)$ we have:*

$$\int_M f^*\omega = deg(f) \int_N \omega.$$

Proof. Let us consider a regular value $q \in N$ of f. We consider the following: $f^{-1}(q) = \{p_1, p_2, \ldots, p_n\}, \quad \epsilon_i = sign(det(df(p_i))), \quad deg(f) = \sum_{i=1}^{n} \epsilon_i.$

There are open neighborhoods $V \subset N$ of q and $U_i \subset M$ of p_i for $i = 1, 2, \ldots, n$ such that:

$i)$ f restricts to a diffeomorphism from $U_i \to V$ $\forall i$; and it is orientation preserving when $\epsilon_i = +1$ and orientation reversing when $\epsilon_i = -1$.

$ii)$The sets U_i are pairwise disjoint.

$iii)f^{-1}(V) = U_1 \cup \ldots \cup U_n$.

Since $df(p_i) : T_{p_i}M \to T_qN$ is a vector space isomorphism it follows from the Inverse Function Theorem that there exists V_i, $i \in \{1, 2, \ldots, n\}$, connected open neighborhoods of q such that $f|U_i : U_i \to V_i$ is a diffeomorphism. We can actually choose U_i to be pairwise disjoint and we have that $f^{-1}(V) = U_1 \cup U_2 \ldots \cup U_n$.

We define:

$$V := V_1 \cap V_2 \cap V_3 \ldots \cap V_n \setminus f(M \setminus U_1 \cup U_2 \ldots \cup U_n).$$

It follows that $U_i \supseteq U_i \cap f^{-1}(V)$. Considering $\omega \in \Omega^m(N)$ with compact support in V we obtain:

$$\int_M f^*\omega = \sum_{i=1}^{n} \int_{U_i} f^*\omega = \sum_{i=1}^{n} \epsilon_i \int_V \omega = deg(f) \int_N \omega.$$

The equations follow from the choice of the sets U_i. We have to assure an universal result (i.e. given another differential m-form ω' with $supp(\omega') \subset V$ and $\int_N \omega = \int_N \omega'$ we will obtain the same equality). This will follow from the fact that ω' is compactly supported and the basic properties of the pull-back with respect to the exterior derivative:

We have that $\int_N (\omega - \omega') = 0$ and using the previous theorem we have that there exists $\tau \in \Omega^{n-1}(N)$ such that $d\tau = \omega - \omega'$. It follows:

$$\begin{aligned}
\int_M f^*\omega &= \int_M f^*(\omega' + d\tau) \\
&= \int_M f^*\omega + \int_M f^*d\tau \\
&= \int_M f^*\omega' + \int_M d(f^*\tau) \\
&= deg(f) \int_N \omega' \\
&= deg(f) \int_N \omega.
\end{aligned}$$

The fact that ω' is compactly supported in V proves the last but one equality and gives the final argument of the proof.

□

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